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Extremely sore on account of the impression having been destroyed. A most transcendent work of this eminent American Mr. Thomas Allen. Americans ought to be proud of the great Specialist. I have many of his Manuscripts in print in this country, though that of his last it would not pay to publish.

The English having acted in all [illegible] of the transcendent Genius of this eminent [illegible] —

APPLICATION

OF THE

ANGULAR ANALYSIS

TO THE SOLUTION OF

INDETERMINATE PROBLEMS

OF THE

SECOND DEGREE.

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PREFACE.

The notation now so universally used for the sine, cosine, &c. of an angle, although in some respects inconvenient, inasmuch as it requires several symbols to designate a quantity where one might be sufficient, is yet so admirably adapted to the wants of Analysis, that it has forced its way into every department of the higher Mathematics. It owes this preference, partly to the power it gives of combining the functions of different angles, or of multiple angles; but principally to its enabling us to avoid the use of the radical sign of the second degree; inasmuch as, if the sine of an angle were represented by one symbol, its cosine would be an irrational function of that symbol.

It would seem strange, then, that this notation has never yet been applied to the Diophantine Analysis, the avowed object of which is to render rational algebraic expressions of certain forms. Perhaps one reason for this may be, that, notwithstanding the attention which has been paid to this branch of Analysis by many mathematicians of the highest rank, the object has been confined to the finding of *numbers* that may fulfil certain conditions, rather than *algebraic forms*, to fulfil those conditions. The methods of solution have been more or less tentative in their character, and could therefore

scarcely be expected to produce results to be compared with the lofty objects of modern Analysis.

In the few pages which follow, I have attempted not only to apply the notation of Trigonometry to several well known Indeterminate Problems of the second degree, but also to introduce somewhat more of system into the method of investigation, and more of generality into the results of the Analysis than has been yet attained. How far I have succeeded, must be left to the judgment of the reader.

In the two last Problems of the second Chapter, I have applied the Analysis to inquiries which would scarcely be attempted without some better method of notation than that hitherto used.

College Point, N. Y.
March 27th, 1848.

ANGULAR ANALYSIS

APPLIED TO

INDETERMINATE EQUATIONS

OF THE SECOND DEGREE.

CHAPTER I.

NOTATION.

THE six trigonometrical functions of an angle A, or of an arc which measures that angle in a circle whose radius is unity, are related to each other as in the following equations:

$$\sin^2 A + \cos^2 A = 1, \qquad \tan A = \frac{\sin A}{\cos A},$$

$$\sin A \operatorname{cosec} A = \cos A \sec A = \tan A \cot A = 1.$$

If, then, the sine and cosine of the angle can be expressed in rational numbers, all the other functions can be so likewise. To do this it is only necessary to take

$$\sin A = \frac{2mn}{m^2+n^2}, \qquad \cos A = \frac{m^2-n^2}{m^2+n^2},$$

which satisfy the first of the above equations, m and n being any rational numbers whatever; then

$$\tan A = \frac{2mn}{m^2 - n^2}, \qquad \cot A = \frac{m^2 - n^2}{2mn},$$

$$\operatorname{cosec} A = \frac{m^2 + n^2}{2mn}, \qquad \sec A = \frac{m^2 + n^2}{m^2 - n^2}.$$

These six equations may all be included in the symbolical form

$$A = \varphi\left(\frac{m}{n}\right), \quad . \quad . \quad . \quad (1)$$

or, in its inverse form

$$\frac{m}{n} = \varphi^{-1}(A); \quad . \quad . \quad . \quad (2)$$

in which, A is any angle or arc,

φ is a functional characteristic,

$\frac{m}{n}$ is the *root of the function.*

For example, when we say that

$$A = \varphi(2),$$

we mean that

$$\sin A = \frac{2 \cdot 2}{2^2 + 1^2} = \tfrac{4}{5}, \quad \cos A = \frac{2^2 - 1^2}{2^2 + 1^2} = \tfrac{3}{5}, \text{ \&c.}$$

Problem I. To find the root of a function, in terms of the angle.

Solution. Take the equation

$$\frac{m^2 - n^2}{m^2 + n^2} = \cos A,$$

and solve it for $\frac{m}{n}$, we shall find

$$\frac{m^2}{n^2} = \frac{1 + \cos A}{1 - \cos A} = \cot^2 \tfrac{1}{2} A,$$

$$\frac{m}{n} = \cot . \tfrac{1}{2} A = \varphi^{-1}(A), \qquad . \quad . \quad (3)$$

so that, *the root of a function is the cotangent of half the angle.*

Cor. While $A = \varphi\left(\frac{m}{n}\right)$ varies from 0° to 180°,

$$\cot \tfrac{1}{2} A = \frac{m}{n}, \text{ varies from } \infty \text{ to } 0,$$

passing through *unity* when $A = 90°$;

thus:

$$0° = \varphi\left(\tfrac{1}{0}\right), \quad 90° = \varphi(1), \quad 180° = \varphi(0), \text{ \&c.}$$

hence also;

$$\text{when } \frac{m}{n} > 1, \qquad \varphi\left(\frac{m}{n}\right) < 90° > 0°;$$

$$\text{when } \frac{m}{n} < 1 > 0, \quad \varphi\left(\frac{m}{n}\right) > 90° < 180°.$$

Again, while $A = \varphi\left(\frac{m}{n}\right)$ varies from 180° to 360°,

$$\cot \tfrac{1}{2} A = \frac{m}{n} \text{ varies from } 0 \text{ to } -\infty,$$

passing through *negative unity* when $A = 270°$; thus:

$$270° = \varphi(-1), \quad 360° = \varphi\left(\tfrac{-1}{0}\right), \text{ \&c.};$$

hence also:

$$\text{when } \frac{m}{n} < 0 > -1, \quad \varphi\left(\frac{m}{n}\right) > 180° < 270°;$$

$$\text{when } \frac{m}{n} < -1 \quad , \quad \varphi\left(\frac{m}{n}\right) > 270° < 360°,$$

&c. &c.

PROBLEM II. Having given the roots of two functions, to find the root of their sum.

Solution. Let $\quad \text{A} + \text{B} = \text{C};$

or if $\quad \text{A} = \varphi\left(\frac{m}{n}\right), \quad \text{B} = \varphi\left(\frac{p}{q}\right), \quad \text{C} = \varphi\left(\frac{r}{s}\right),$

$$\varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{q}\right) = \varphi\left(\frac{r}{s}\right).$$

But we have

$$\cot \tfrac{1}{2}\text{C} = \cot \tfrac{1}{2}(\text{A} + \text{B}) = \frac{\cot \frac{1}{2}\text{A} \cot \frac{1}{2}\text{B} - 1}{\cot \frac{1}{2}\text{A} + \cot \frac{1}{2}\text{B}},$$

or,

$$\frac{r}{s} = \frac{\frac{m}{n} \cdot \frac{p}{q} - 1}{\frac{m}{n} + \frac{p}{q}} = \frac{mp - nq}{mq + np};$$

and the above equation becomes

$$\varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{q}\right) = \varphi\left(\frac{mp - nq}{mq + np}\right). \quad . \quad . \quad (4)$$

Example 1. $90° + \varphi\left(\frac{m}{n}\right) = \varphi(1) + \varphi\left(\frac{m}{n}\right) = \varphi\left(\frac{m-n}{m+n}\right)$; (5)

thus: $\quad 90° + \varphi(2) = \varphi(\tfrac{1}{3}),$

$\quad 90° + \varphi(\tfrac{3}{2}) = \varphi(\tfrac{1}{5}),$

&c.

Example 2. $180^\circ+\varphi\left(\frac{m}{n}\right)=\varphi(0)+\varphi\left(\frac{m}{n}\right)=\varphi\left(-\frac{n}{m}\right)$ (6)

thus : $\quad 180^\circ+\varphi\ (2)\ =\varphi(-\frac{1}{2})$, &c.

Cor. 1. Let, in equation (4),

$$c=\varphi\left(\frac{mp-nq}{mq+np}\right)=0,$$

then, by Prob. I., Cor.

$$mq+np=0, \quad \text{and } \frac{p}{q}=-\frac{m}{n};$$

so that the equation becomes

$$\varphi\left(\frac{m}{n}\right)+\varphi\left(-\frac{m}{n}\right)=0,$$

$$\text{or} \quad \varphi\left(-\frac{m}{n}\right)=-\varphi\left(\frac{m}{n}\right). \quad . \quad . \quad (7)$$

Cor. 2. Let, in equation (4), $\frac{p}{q}=\frac{m}{n}$; then

$$2\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{m}{n}\right)+\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{m^2-n^2}{2mn}\right). \quad . \quad (8)$$

In like manner,

$$3\varphi\left(\frac{m}{n}\right)=2\varphi\left(\frac{m}{n}\right)+\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{m^2-n^2}{2mn}\right)+\varphi\left(\frac{m}{n}\right)$$

$$=\varphi\left\{\frac{(m^2-n^2)m-2mn^2}{(m^2-n^2)n+2m^2n}\right\}=\varphi\left\{\frac{m}{n}\cdot\frac{m^2-3n^2}{3m^2-n^2}\right\}; \quad (9)$$

$$4\varphi\left(\frac{m}{n}\right)=2\,.\,2\varphi\left(\frac{m}{n}\right)=2\varphi\left(\frac{m^2-n^2}{2mn}\right)$$

$$=\varphi\left\{\frac{(m^2-n^2)^2-4m^2n^2}{4mn(m^2-n^2)}\right\}=\varphi\left\{\frac{m^4-6m^2n^2+n^4}{4mn(m^2-n^2)}\right\};(10)$$

&c. &c.

For example :

$$2\varphi(2)=\varphi(\tfrac{5}{4}),\quad 3\varphi(2)=\varphi(\tfrac{2}{11}),\quad 4\varphi(2)=\varphi(-\tfrac{7}{24}),\ \&c.$$

$$2\varphi(\tfrac{3}{2})=\varphi(\tfrac{5}{12}),\quad 3\varphi(\tfrac{3}{2})=\varphi(-\tfrac{9}{46}),\quad 4\varphi(\tfrac{3}{2})=\varphi(-\tfrac{119}{60}),\ \&c.$$

Problem III. Having given the roots of two functions, to find the root of their difference.

Solution. Write, in equation (4), $-m$ for m, and by (7)

$$\varphi\left(\frac{p}{q}\right)-\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{p}{q}\right)+\varphi\left(-\frac{m}{n}\right)=\varphi\left(\frac{mp+nq}{mq-np}\right). \quad (11)$$

Example 1. $$90^\circ-\varphi\left(\frac{m}{n}\right)=\varphi(1)-\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{m+n}{m-n}\right). \quad (12)$$

Thus, $\varphi(2)+\varphi(3)=\varphi(4)+\varphi(\tfrac{5}{3})=\varphi(5)+\varphi(\tfrac{3}{2})=\&c.=90^\circ.$

Example 2. $$180^\circ-\varphi\left(\frac{m}{n}\right)=\varphi(0)-\varphi\left(\frac{m}{n}\right)=\varphi\left(\frac{n}{m}\right). \quad (13)$$

Thus, $\varphi(2)+\varphi(\tfrac{1}{2})=\varphi(\tfrac{2}{3})+\varphi(\tfrac{3}{2})=\varphi(4)+\varphi(\tfrac{1}{4})=\&c.=180^\circ.$

Cor. 1. $$180^\circ-\varphi\left(\frac{m}{n}\right)-\varphi\left(\frac{p}{q}\right)=180^\circ-\varphi\left(\frac{mp-nq}{mq+np}\right)=\varphi\left(\frac{mq+np}{mp-nq}\right),$$

or $$\varphi\left(\frac{m}{n}\right)+\varphi\left(\frac{p}{q}\right)+\varphi\left(\frac{mq+np}{mp-nq}\right)=180^\circ, \quad . \quad (14),$$

and therefore

$$\varphi\left(\frac{m}{n}\right),\quad \varphi\left(\frac{p}{q}\right),\quad \varphi\left(\frac{mq+np}{mp-nq}\right)$$

represent ***the three angles of a plane triangle***, the functions of

each of which are expressed in rational numbers. For example,

$$\varphi(2),\ \varphi(\tfrac{3}{2}),\ \varphi(\tfrac{7}{4});$$

$$\varphi(3),\ \varphi(4),\ \varphi(\tfrac{7}{11});\ \&c.$$

are the angles of a plane triangle, and their trigonometrical functions can all be expressed in rational numbers.

Cor. 2. $$90^\circ - \varphi\left(\frac{m}{n}\right) - \varphi\left(\frac{p}{q}\right) = \varphi(1) - \varphi\left(\frac{mp-nq}{mq+np}\right)$$

$$= \varphi\left\{\frac{m(p+q)+n(p-q)}{m(p-q)-n(p+q)}\right\},$$

or, $$\varphi\left(\frac{m}{n}\right) + \varphi\left(\frac{p}{q}\right) + \varphi\left\{\frac{m(p+q)+n(p-q)}{m(p-q)-n(p+q)}\right\} = 90^\circ, \quad (15)$$

Thus $\varphi(2) + \varphi(\frac{3}{2}) + \varphi(-\frac{11}{3}) = \varphi(3) + \varphi(4) + \varphi(\frac{9}{2}) = 90^\circ.$

Scholium. The results of this Chapter show that

1°. If the cotangent or tangent of half an angle is expressed by a rational number, all the trigonometrical functions of that angle will be expressed by rational numbers.

2°. If the trigonometrical functions of any angle can be expressed in rational numbers, the functions of any multiple of that angle can be also expressed in rational numbers.

3°. If the trigonometrical functions of any number of angles can be expressed in rational numbers, the functions of the angles, produced by adding or subtracting these angles in any manner whatever, can also be expressed in rational numbers.

CHAPTER II.

APPLICATION OF THE PRECEDING NOTATION TO INDETERMINATE EQUATIONS OF THE SECOND DEGREE.

PROBLEM I. To divide a given square number (a^2) into two other square numbers.

Solution. The equation to be solved is

$$x^2 + y^2 = a^2,$$

so that if we take

$$x = a \sin A, \qquad y = a \cos A;$$

we shall have

$$x^2 + y^2 = a^2(\sin^2 A + \cos^2 A) = a^2;$$

and A may be any angle whose functions are rational numbers.

If a be itself the sum of two squares; or

$$a = m^2 + n^2,$$

by taking $A = \varphi\left(\frac{m}{n}\right)$, we shall find integral values for x and y. For example, if

$$a = 65 = 8^2 + 1^2 = 7^2 + 4^2 = (2^2 + 1^2).13 = (3^2 + 2^2).5;$$

we may take for A the several values

$$\varphi(2), \quad \varphi(\tfrac{3}{2}), \quad \varphi(\tfrac{7}{4}), \quad \varphi(8);$$

and we shall find for x the several values

$$52, \quad 60, \quad 56, \quad 16;$$

and the corresponding values of y are

$$39, \quad 25, \quad 33, \quad 63.$$

Thus $65^2 = 52^2 + 39^2 = 60^2 + 25^2 = 56^2 + 33^2 = 16^2 + 63^2$.

Cor. The given equation may be written

$$a^2 - x^2 = y^2,$$

which may serve to divide a given square number (y^2) into the difference of two squares. By solving the above equations for a and x we find

$$a = y \sec \text{A}, \qquad x = a \sin \text{A} = y \tan \text{A};$$

in fact,

$$a^2 - x^2 = y^2(\sec^2 \text{A} - \tan^2 \text{A}) = y^2.$$

It must be recollected that, since A may have any value, we can write its complement instead of it; that is we may take, as well,

$$a = y \operatorname{cosec} \text{A}, \qquad x = y \cot \text{A}.$$

Example. If we take $\text{A} = \varphi(y)$, we have

$$a = \tfrac{1}{2}(y^2 + 1), \qquad x = \tfrac{1}{2}(y^2 - 1).$$

Problem II. To divide the sum of two given squares ($a^2 + b^2$) into two other square numbers.

Solution. The equation to be solved is

$$x^2 + y^2 = a^2 + b^2.$$

Then we may take

$$x = a \cos \text{A} + b \sin \text{A}, \qquad y = a \sin \text{A} - b \cos \text{A};$$

for then,

$$x^2+y^2=a^2(\cos^2 A+\sin^2 A)+ab(\sin 2A-\sin 2A)$$
$$+b^2(\sin^2 A+\cos^2 A)=a^2+b^2.$$

Example. If $A=\varphi(2)$, then

$$x=\tfrac{1}{5}(3a+4b), \quad y=\tfrac{1}{5}(4a-3b).$$

Cor. 1. If $a=b$, or the equation is

$$x^2+y^2=2a^2,$$

the above values become

$$x=a(\cos A+\sin A)=a\sqrt{2}\cos(A-45^\circ),$$
$$y=a(\sin A-\cos A)=a\sqrt{2}\sin(A-45^\circ);$$

and x, y, a are the roots of *three square numbers in arithmetical progression*, a^2 being the mean.

If $A=\varphi\left(\frac{m}{n}\right)$, these roots will be of the usual form,

$$x=m^2+2mn-n^2, \quad a=m^2+n^2, \quad y=-m^2+2mn+n^2.$$

Cor. 2. If the given equation be written in the form

$$x^2-a^2=b^2-y^2,$$

it will serve to divide the difference of two given squares into the difference of two other squares, by solving the preceding equations for x and a; viz.

$$x=b\operatorname{cosec} A+y\cot A, \qquad a=b\cot A+y\operatorname{cosec} A;$$

in fact

$$x^2-a^2=b^2(\operatorname{cosec}^2 A-\cot^2 A)+2by(\cot A\operatorname{cosec} A-\cot A\operatorname{cosec} A)$$
$$+y^2(\cot^2 A-\operatorname{cosec}^2 A)=b^2-y^2.$$

Cor. 3. In all these cases we can write $90^\circ \pm A$, or $180^\circ \pm A$ for A, so that

$$(a\cos A \pm b\sin A)^2 + (a\sin A \mp b\cos A)^2 = a^2 + b^2,$$

$$\{a\surd 2\cos(45^\circ \pm A)\} + \{a\surd 2\sin(45^\circ \pm A)\}^2 = 2a^2,$$

$$(b\sec A \pm y\tan A)^2 - (b\tan A \pm y\sec A)^2 = b^2 - y^2.$$

PROBLEM III. To find two square numbers, whose difference shall be equal to a given number ($a = bc$).

Solution. The equation to be solved is

$$x^2 - y^2 = a = bc,$$

and will be satisfied by taking

$$x + y = b\cot\tfrac{1}{2}A, \qquad x - y = c\tan\tfrac{1}{2}A.$$

Then, by addition and subtraction,

$$x = \tfrac{1}{2}(b\cot\tfrac{1}{2}A + c\tan\tfrac{1}{2}A) = \frac{b\cos^2\frac{1}{2}A + c\sin^2\frac{1}{2}A}{\sin A}$$

$$= \tfrac{1}{2}(b+c)\operatorname{cosec} A + \tfrac{1}{2}(b-c)\cot A,$$

$$y = \tfrac{1}{2}(b\cot\tfrac{1}{2}A - c\tan\tfrac{1}{2}A) = \frac{b\cos^2\frac{1}{2}A - c\sin^2\frac{1}{2}A}{\sin A}$$

$$= \tfrac{1}{2}(b-c)\operatorname{cosec} A + \tfrac{1}{2}(b+c)\cot A.$$

Example. Let $a = 15 = 5 . 3$, we may have

$$x = 8\operatorname{cosec} A + 7\cot A, \qquad y = 7\operatorname{cosec} A + 8\cot A;$$

or, $\quad x = 4\operatorname{cosec} A + \cot A, \qquad y = \operatorname{cosec} A + 4\cot A.$

Thus, if $A = 90^\circ$, $x = 8$ or 4, $y = 7$ or 1,

if $A = \varphi(2)$, $x = \frac{61}{4}$ or $\frac{23}{4}$, $y = \frac{59}{4}$ or $\frac{17}{4}$; &c.

and $15 = 4^2 - 1^2 = 8^2 - 7^2 = (\frac{23}{4})^2 - (\frac{17}{4})^2 = (\frac{61}{4})^2 - (\frac{59}{4})^2$, &c.

Cor. If a has the form $k(t^2 - u^2)$, or the given equation is

$$x^2 - y^2 = k(t^2 - u^2),$$

we may take $b = k(t + u)$, $c = t - u$; and the above values become

$$x \sin A = (k \cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A)t + (k \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)u,$$

$$y \sin A = (k \cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)t + (k \cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A)u.$$

Or, we may take $b = t + u$, $c = k(t - u)$; then

$$x \sin A = (\cos^2 \tfrac{1}{2}A + k \sin^2 \tfrac{1}{2}A)t + (\cos^2 \tfrac{1}{2}A - k \sin^2 \tfrac{1}{2}A)u,$$

$$y \sin A = (\cos^2 \tfrac{1}{2}A - k \sin^2 \tfrac{1}{2}A)t + (\cos^2 \tfrac{1}{2}A + k \sin^2 \tfrac{1}{2}A)u.$$

Thus if $k = 2$, we shall have, to solve the equation

$$x^2 - y^2 = 2(t^2 - u^2),$$

the values of x and y,

$$x \sin A = (1 + \cos^2 \tfrac{1}{2}A)t + (3 \cos^2 \tfrac{1}{2}A - 1)u,$$

$$y \sin A = (3 \cos^2 \tfrac{1}{2}A - 1)t + (1 + \cos^2 \tfrac{1}{2}A)u;$$

or,

$$x \sin A = (1 + \sin^2 \tfrac{1}{2}A)t + (1 - 3 \sin^2 \tfrac{1}{2}A)u,$$

$$y \sin A = (1 - 3 \sin^2 \tfrac{1}{2}A)t + (1 + \sin^2 \tfrac{1}{2}A)u.$$

PROBLEM IV. To solve the equation, in rational numbers,

$$x^2 + axy + by^2 = z^2,$$

in which a and b are given numbers.

Solution. This equation is the same as

$$z^2 - x^2 = axy + by^2 = y(ax + by);$$

and we may take, as in Problem III.,

$$z + x = y \cot \tfrac{1}{2}A,$$

$$z - x = (ax + by) \tan \tfrac{1}{2}A;$$

and eliminating z,

$$2x = y(\cot \tfrac{1}{2}A - b \tan \tfrac{1}{2}A) - ax \tan \tfrac{1}{2}A,$$

or, $$(\sin A + a \sin^2 \tfrac{1}{2}A)x = (\cos^2 \tfrac{1}{2}A - b \sin^2 \tfrac{1}{2}A)y.$$

So that we can make

$$x = t(\cos^2 \tfrac{1}{2}A - b \sin^2 \tfrac{1}{2}A),$$

$$y = t(\sin A \quad + a \sin^2 \tfrac{1}{2}A),$$

$$z = t(\cos^2 \tfrac{1}{2}A + \tfrac{1}{2}a \sin A + b \sin \tfrac{1}{2}A);$$

in which t is any rational number whatever. Thus, if we take $A = \varphi\left(\frac{m}{n}\right)$, and $t = m^2 + n^2$, we have

$$x = m^2 - bn^2,$$

$$y = 2mn + an^2,$$

$$z = m^2 + amn + bn^2.$$

Example 1. Let $a = 1$, $b = 1$; the equation is

$$x^2 + xy + y^2 = z^2;$$

and it is solved by taking

$$x = t \cos A, \quad y = t(\sin A + \sin^2 \tfrac{1}{2}A), \quad z = t\,(1 + \tfrac{1}{2} \sin A).$$

Thus, if $A = \varphi(2)$, $t = 5$; we have $x = 3$, $y = 5$, $z = 7$.

Example 1. Let $a = 1$, the equation is

$$x^2 + 7y^2 = z^2;$$

and it is solved by taking

$$x = [illegible], \quad y = [illegible] \sin A, \quad z = [illegible].$$

PROBLEM [illegible]. To solve the two equations

$$x^2 + ax = y^2,$$

$$x^2 - ax = z^2,$$

a being a given number.

Solution. By adding the given equations we find

$$2x^2 = y^2 + z^2;$$

and therefore, by Prob. [illegible], Cor. [illegible],

$$y = x\sqrt{2}\sin(45^\circ + A), \qquad z = x\sqrt{2}\cos(45^\circ + A).$$

By subtracting the given equations, we have

$$2ax = y^2 - z^2 = 2x^2\{\sin^2(45^\circ + A) - \cos^2(45^\circ + A)\}$$

$$= -2x^2\cos(90^\circ + 2A) = 2x^2\sin 2A,$$

and

$$x = \frac{a}{\sin 2A}.$$

Example 1. If a is of the form $m^2 - n^2$, we may take $A = \phi\left(\frac{m}{n}\right)$, and then

$$x = [illegible].$$

Example 2. If $a = [illegible]$, and we take $A = \phi([illegible])$, then

$$x = [illegible].$$

PROBLEM VI. To find three numbers in arithmetical progression, such that the sums of every two of them may be square numbers.

Solution. The equations to be solved are

$$x + y = a^2, \qquad y + z = c^2,$$

$$x + z = b^2, \qquad x + z = 2y.$$

By eliminating x, y, z, we find

$$a^2 + c^2 = 2b^2,$$

and therefore, by Prob. II., Cor. 1,

$$a = b(\sin A + \cos A,) \qquad c = b(\sin A - \cos A);$$

hence we have

$$x = \tfrac{1}{2}(2a^2 - b^2) = \tfrac{1}{2}b^2(1 + 2\sin 2A),$$

$$y = \tfrac{1}{2}b^2,$$

$$z = \tfrac{1}{2}(2c^2 - b^2) = \tfrac{1}{2}b^2(1 - 2\sin 2A).$$

To render these results positive, it is necessary that

$$\sin 2A < \tfrac{1}{2}, \qquad 2A < 30°,$$

$$\cot \tfrac{1}{2}A = \varphi^{-1}(A) > 7{,}604 \ldots\ldots$$

Example. If $A = \varphi(9)$, $2A = \varphi(\frac{40}{9})$, $\sin 2A = \frac{720}{1681}$, and taking $b = 82$, we have

$$x = 6242, \qquad y = 3362, \qquad z = 482.$$

PROBLEM. VII. To find four numbers, such that the sum of every two of them may be a square number.

Solution. The equations to be solved are

$$v+x=a^2, \qquad x+y=d^2,$$

$$v+y=b^2, \qquad x+z=e^2,$$

$$v+z=c^2, \qquad y+z=f^2;$$

and they give

$$2v= \quad a^2+b^2-d^2, \qquad 2x=a^2-b^2+d^2,$$

$$2y=-a^2+b^2+d^2, \qquad 2z=c^2+f^2-b^2.$$

$$v+x+y+z=a^2+f^2=b^2+e^2=c^2+d^2,$$

These last equations may be solved by Prob. II., and give

$$e=d\cos \text{A}+c\sin \text{A}, \qquad b=d\sin \text{A}-c\cos \text{A},$$

$$f=d\cos \text{B}+c\sin \text{B}, \qquad a=d\sin \text{B}-c\cos \text{B}.$$

By substitution we find

$$2v=d^2(\sin^2\text{A}-\cos^2\text{B})-cd(\sin 2\text{A}+\sin 2\text{B})+c^2(\cos^2\text{A}+\cos^2\text{B}),$$

$$2x=d^2(\sin^2\text{B}+\cos^2\text{A})+cd(\sin 2\text{A}-\sin 2\text{B})+c^2(\cos^2\text{B}-\cos^2\text{A}),$$

$$2y=d^2(\sin^2\text{A}+\cos^2\text{B})-cd(\sin 2\text{A}-\sin 2\text{B})+c^2(\cos^2\text{A}-\cos^2\text{B}),$$

$$2z=d^2(\cos^2\text{B}-\sin^2\text{A})+cd(\sin 2\text{A}+\sin 2\text{B})+c^2(\sin^2\text{A}+\sin^2\text{B}).$$

These values must contain every possible solution of the question, but in order to have positive numbers, it is necessary that we should have, not only

$$\left\{\frac{d}{c}\cos(\text{A}+\text{B})+\sin(\text{A}+\text{B})\right\}^2<1+\frac{\cos(\text{A}+\text{B})}{\cos(\text{A}-\text{B})}>1-\frac{\cos(\text{A}+\text{B})}{\cos(\text{A}-\text{B})}:$$

but also, supposing $\text{A}>\text{B}$,

$$\left\{\frac{c}{d}\sin(\text{A}+\text{B})+\cos(\text{A}+\text{B})\right\}^2<1+\frac{\sin(\text{A}+\text{B})}{\sin(\text{A}-\text{B})}>1-\frac{\sin(\text{A}+\text{B})}{\sin(\text{A}-\text{B})}.$$

If we take the particular case $B = 90° - A$, we have

$$2v = c^2 - 2cd \sin 2A \qquad 2x = 2d^2 \cos^2 A - c^2 \cos 2A,$$

$$2y = 2d^2 \sin^2 A + c^2 \cos 2A, \quad 2z = c^2 + 2cd \sin 2A,$$

and the limits for positive numbers reduce to

$$\frac{c}{d} > 2 \sin 2A, \qquad \text{and} \left(\frac{c}{d}\right)^2 < 1 + \sec 2A > 1 - \sec 2A.$$

Example. If $A = \varphi(\frac{5}{2})$, $B = 90° - A$, the limits become

$$\frac{c}{d} > 2 \times \tfrac{840}{841} < 4, 64 \ldots;$$

by taking $c = 58$, $d = 29$, the numbers will be

2, 359, 482, 3362.

In the manuscripts bequeathed to me by my lamented friend, William Lenhart, Esq., York, Penn., I find the following numbers for this question,

18, 882, 2482, 4743;

which are probably smaller than any previously found. They may be obtained from this solution by putting

$$d = 75, \quad c = 50, \quad A = \varphi(\tfrac{1}{7}), \quad B = \varphi(2).$$

PROBLEM VIII. To find two numbers, such that if unity be added to each of them, as also to their sum and difference, the four sums will be square numbers.

Solution. The equations to be solved are

$$x + 1 = a^2, \qquad x + y + 1 = c^2,$$

$$y + 1 = b^2, \qquad x - y + 1 = d^2.$$

By eliminating x and y, they are

$$a^2 + b^2 - 1 = c^2, \qquad a^2 - b^2 + 1 = d^2;$$

or by addition and subtraction,

$$2a^2 = c^2 + d^2, \qquad b^2 - 1 = \tfrac{1}{2}(c^2 - d^2).$$

The first of these gives, by Prob. II., Cor. 1,

$$c = a(\sin A + \cos A), \qquad d = a(\sin A - \cos A);$$

and the second becomes

$$b^2 - 1 = a^2 \sin 2A.$$

Hence we get, by putting, as in Prob. III.,

$$b + 1 = a \cot \tfrac{1}{2}B, \qquad b - 1 = a \sin 2A \tan \tfrac{1}{2}B;$$

$$b \sin B = a(\cos^2 \tfrac{1}{2}B + \sin 2A \sin^2 \tfrac{1}{2}B,$$

$$\sin B = a(\cos^2 \tfrac{1}{2}B - \sin 2A \sin^2 \tfrac{1}{2}B);$$

or $$a = \frac{\sin B}{\cos^2 \frac{1}{2}B - \sin 2A \sin^2 \frac{1}{2}B},$$

$$b = \frac{\cos^2 \frac{1}{2}B + \sin 2A \sin^2 \frac{1}{2}B}{\cos^2 \frac{1}{2}B - \sin 2A \sin^2 \frac{1}{2}B};$$

and the required numbers will be

$$x = a^2 - 1 \text{ and } y = b^2 - 1.$$

These equations contain all the solutions of the question; but to obtain whole numbers, some modification is necessary.

Take $B = \varphi\left(\frac{t}{u}\right)$, then

$$a = \frac{2tu}{t^2 - \sin 2Au^2}, \qquad b = \frac{t^2 + \sin 2Au^2}{t^2 - \sin 2Au^2},$$

and for every value of A a series of whole numbers can be obtained for a and b by the methods of Lagrange or Gauss.

Thus if $A=\varphi(2)$, $t=2r$, $u=5s$, these equations become

$$a=\frac{5rs}{r^2-6s^2}, \qquad b=\frac{r^2+6s^2}{r^2-6s^2},$$

then if we take

$$r=\tfrac{1}{2}\{(5+2\sqrt{6})^i+(5-2\sqrt{6})^i\},\ s=\frac{1}{2\sqrt{6}}\{(5+2\sqrt{6})^i-(5-2\sqrt{6})^i\},$$

i being any integer, we shall always have

$$r^2-6s^2=1,$$

and therefore a and b integers. For $i=1$, we have $r=5$, $s=2$; $a=50$, $b=49$; $x=2499$, $y=2400$.

Problem IX. To solve the equations, in rational numbers,

$$\mathbf{f}(z,y,x\ldots)=a^2+z^2=b^2+y^2=c^2+x^2=\&c.$$

in which $\mathbf{f}$ is a functional characteristic.

Solution. Take, as in Problem II.,

$$b=a\cos A+z\sin A, \qquad y=a\sin A-z\cos A,$$

$$c=a\cos B+z\sin B, \qquad x=a\sin B-z\cos B,$$

$$\&c. \qquad\qquad \&c.$$

and we shall thus have

$$a^2+z^2=b^2+y^2=c^2+x^2=\&c.;$$

and the values of y, x, &c., are to be written in the equation

$$\mathbf{f}(z,y,x\ldots.)=a^2+z^2, \quad . \quad . \quad (a)$$

and this equation must be solved for z.

Thus, if $A=90°$, $x=8$ or 4, $y=7$ or 1,

if $A=\varphi(2)$, $x=\frac{61}{4}$ or $\frac{23}{4}$, $y=\frac{59}{4}$ or $\frac{17}{4}$; &c.

and $15=4^2-1^2=8^2-7^2=(\frac{23}{4})^2-(\frac{17}{4})^2=(\frac{61}{4})^2-(\frac{59}{4})^2$, &c.

Cor. If a has the form $k(t^2-u^2)$, or the given equation is

$$x^2-y^2=k(t^2-u^2),$$

we may take $b=k(t+u)$, $c=t-u$; and the above values become

$$x \sin A=(k \cos^2 \tfrac{1}{2}A+\sin^2 \tfrac{1}{2}A)t+(k \cos^2 \tfrac{1}{2}A-\sin^2 \tfrac{1}{2}A)u,$$

$$y \sin A=(k \cos^2 \tfrac{1}{2}A-\sin^2 \tfrac{1}{2}A)t+(k \cos^2 \tfrac{1}{2}A+\sin^2 \tfrac{1}{2}A)u.$$

Or, we may take $b=t+u, c=k(t-u)$; then

$$x \sin A=(\cos^2 \tfrac{1}{2}A+k \sin^2 \tfrac{1}{2}A)t+(\cos^2 \tfrac{1}{2}A-k \sin^2 \tfrac{1}{2}A)u,$$

$$y \sin A=(\cos^2 \tfrac{1}{2}A-k \sin^2 \tfrac{1}{2}A)t+(\cos^2 \tfrac{1}{2}A+k \sin^2 \tfrac{1}{2}A)u.$$

Thus if $k=2$, we shall have, to solve the equation

$$x^2-y^2=2(t^2-u^2),$$

the values of x and y,

$$x \sin A=(1+\cos^2 \tfrac{1}{2}A)t+(3 \cos^2 \tfrac{1}{2}A-1)u,$$
$$y \sin A=(3 \cos^2 \tfrac{1}{2}A-1)t+(1+\cos^2 \tfrac{1}{2}A)u;$$

or,
$$x \sin A=(1+\sin^2 \tfrac{1}{2}A)t+(1-3 \sin^2 \tfrac{1}{2}A)u,$$
$$y \sin A=(1-3 \sin^2 \tfrac{1}{2}A)t+(1+\sin^2 \tfrac{1}{2}A)u.$$

PROBLEM IV. To solve the equation, in rational numbers,

$$x^2+axy+by^2=z^2,$$

in which a and b are given numbers.

Solution. This equation is the same as

$$z^2 - x^2 = axy + by^2 = y(ax + by);$$

and we may take, as in Problem III.,

$$z + x = y \cot \tfrac{1}{2}A,$$
$$z - x = (ax + by) \tan \tfrac{1}{2}A;$$

and eliminating z,

$$2x = y(\cot \tfrac{1}{2}A - b \tan \tfrac{1}{2}A) - ax \tan \tfrac{1}{2}A,$$

or, $$(\sin A + a \sin^2 \tfrac{1}{2}A)x = (\cos^2 \tfrac{1}{2}A - b \sin^2 \tfrac{1}{2}A)y.$$

So that we can make

$$x = t(\cos^2 \tfrac{1}{2}A - b \sin^2 \tfrac{1}{2}A),$$
$$y = t(\sin A \quad + a \sin^2 \tfrac{1}{2}A),$$
$$z = t(\cos^2 \tfrac{1}{2}A + \tfrac{1}{2}a \sin A + b \sin \tfrac{1}{2}A);$$

in which t is any rational number whatever. Thus, if we take $A = \varphi\left(\frac{m}{n}\right)$, and $t = m^2 + n^2$, we have

$$x = m^2 - bn^2,$$
$$y = 2mn + an^2,$$
$$z = m^2 + amn + bn^2.$$

Example 1. Let $a = 1$, $b = 1$; the equation is

$$x^2 + xy + y^2 = z^2;$$

and it is solved by taking

$$x = t \cos A, \quad y = t(\sin A + \sin^2 \tfrac{1}{2}A), \quad z = t\,(1 + \tfrac{1}{2} \sin A).$$

Thus, if $A = \varphi(2)$, $t = 5$; we have $x = 3$, $y = 5$, $z = 7$.

General Case. Let

$$f(z, y, x \ldots\ldots) = (z + y + x + \&c.)^r = s^2,$$

or,
$$z + y + x + \&c. = s^{\frac{2}{r}},$$

and equation (a) becomes

$$\cot A + \cot B + \cot C + \&c. = \Sigma \cot A = s^{\frac{2}{r}-1}.$$

But since $\cot A = \frac{1}{2}(\cot \frac{1}{2}A - \tan \frac{1}{2}A)$,

$$\Sigma \cot \tfrac{1}{2}A - \Sigma \tan \tfrac{1}{2}A = 2s^{\frac{2}{r}-1}.$$

Now put
$$\cot \tfrac{1}{2}B = k \tan \tfrac{1}{2}A,$$
$$\cot \tfrac{1}{2}C = l \tan \tfrac{1}{2}A,$$
$$\&c.,$$

and the preceding equation becomes

$$\cot \tfrac{1}{2}A\left(1 - \Sigma \frac{1}{k}\right) + \tan \tfrac{1}{2}A(\Sigma k - 1) = 2s^{\frac{2}{r}-1}.$$

Now let k, l, m, &c., be taken so that

$$\Sigma \frac{1}{k} = 1.$$

and we shall have

$$\cot \tfrac{1}{2}A = \tfrac{1}{2}(\Sigma k - 1)s^{1-\frac{2}{r}}, \quad . \quad . \quad (b)$$

which completely determines the problem.

This class of questions was, I believe, devised by my early friend, Mr. T. Beverley, who died about the year 1833; and this is very nearly his solution, translated into the notation of the angular analysis. The only one of his questions to which I now have access is question 641, No. 28, Math. Companion, viz.

"Find numbers, *ad libitum*, whose sum is a $4n$th power,

and such that, if the square of each be added to their sum, the several sums shall be all squares."

In this case we must have

$$r = 1, \quad s^2 = t^{4n}, \quad s = t^{2n},$$

$$\cot \tfrac{1}{2}A = \tfrac{1}{2}(\Sigma k - 1)t^{-2n},$$

and to render the numbers positive, it is only necessary to have

$$\cot \tfrac{1}{2}A > 1 < k,$$

$$\text{or} \qquad t^{2n} < \frac{\Sigma k - 1}{2} > \frac{\Sigma k - 1}{2k},$$

k being the least of the numbers k, l, m, &c.

Example. To find four numbers, whose sum is a biquadrate, or $n = 1$, having the required properties; then

$$\cot \tfrac{1}{2}A = \tfrac{1}{2}(\Sigma k - 1)t^{-2},$$

Let $k = 2$, $l = 3$, $m = 6$, then $\Sigma \dfrac{1}{k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{6} = 1$, and $\Sigma k = 2 + 3 + 6 = 11$, so that we must have

$$t^2 < 5 > 2\tfrac{1}{2}.$$

If $t = 2$, we get $s = 4$, $\cot \frac{1}{2}A = \frac{5}{4}$, $\cot \frac{1}{2}B = \frac{8}{5}$, $\cot \frac{1}{2}C = \frac{12}{5}$, $\cot \frac{1}{2}D = \frac{24}{5}$, and the numbers are

$$z = \tfrac{9}{10}, \quad y = \tfrac{39}{20}, \quad x = \tfrac{119}{30}, \quad v = \tfrac{551}{60},$$

whose sum is 16.

Scholium. The functions indicated by (f) in the four preceding problems, are understood to be not only rational and algebraic, but also such, in general, as to render the final equations (*a*) of not more than the second degree. These

equations will then generally be dependent, as will be shown in many of the following examples, on another equation of the form

$$k^2 = a + b\cos A + c\sin A + d\sin A\cos A + e\sin^2 A,$$

which, for the purpose of solution, may be put in the form shown in the next Problem, to which it is easily reduced.

PROBLEM XIII. To solve the equation, in rational numbers,

$$k^2 = m\cos^2\tfrac{1}{2}A + n\sin^2\tfrac{1}{2}A + (p\cos^2\tfrac{1}{2}A + q\sin^2\tfrac{1}{2}A)\sin A + r\sin^2 A.$$

Solution. Since the root of this square cannot have a term containing $\sin\frac{1}{2}A$ and $\cos\frac{1}{2}A$ in an odd degree, it must be of the form

$$\pm k = d\cos^2\tfrac{1}{2}A + e\sin^2\tfrac{1}{2}A + f\sin A,$$

$$k^2 = d^2\cos^4\tfrac{1}{2}A + e^2\sin^4\tfrac{1}{2}A$$

$$+ 2f\sin A(d\cos^2\tfrac{1}{2}A + e\sin^2\tfrac{1}{2}A) + (f^2 + \tfrac{1}{2}de)\sin^2 A;$$

and to make this value of k^2 equal to that in the problem, it is necessary that

$$(m - d^2\cos^2\tfrac{1}{2}A)\cos^2\tfrac{1}{2}A + (n - e^2\sin^2\tfrac{1}{2}A)\sin^2\tfrac{1}{2}A$$

$$+ \{(p - 2df)\cos^2\tfrac{1}{2}A + (q - 2ef)\sin^2\tfrac{1}{2}A\}\sin A$$

$$+ (r - f^2 - \tfrac{1}{2}de)\sin^2 A = 0. \quad . \ (a)$$

There are, in general, three cases in which this equation can be completely resolved; that is, in which the value of $\tan\frac{1}{2}A$ or $\cot\frac{1}{2}A$ can be expressed in a rational form.

Case 1. When $m = \mu^2$ and $n = \nu^2$ are complete squares; then we may take

$$d^2 = \mu^2, \quad e^2 = \nu^2; \qquad \text{or } d = \mu, \quad e = \pm\nu,$$

then, substituting these in equation (a), and dividing it by $\sin \text{A}$, it becomes

$$(p-2\mu f)\cos^2\tfrac{1}{2}\text{A}+(q\mp 2\nu f)\sin^2\tfrac{1}{2}\text{A}+\{r-f^2+\tfrac{1}{4}(\mu\mp\nu)^2\}\sin\text{A}=0.$$

We may now take either, first,

$$p-2\mu f=0, \qquad f=\frac{p}{2\mu};$$

$$\tan\tfrac{1}{2}\text{A}=\frac{2r-2f^2+\frac{1}{2}(\mu\mp\nu)^2}{\pm 2\nu f-q}=\frac{4r\mu^2+(\mu\mp\nu)^2\mu^2-p^2}{2\mu(\pm p\nu-q\mu)}; \quad (b)$$

and the root of the square is

$$\pm k=\mu\cos^2\tfrac{1}{2}\text{A}\pm\nu\sin^2\tfrac{1}{2}\text{A}+\frac{p}{2\mu}\sin\text{A}. \quad . \quad . \quad (c)$$

Or, secondly, we may take

$$q\mp 2\nu f=0, \qquad f=\pm\frac{q}{2\nu};$$

$$\cot\tfrac{1}{2}\text{A}=\frac{2r-2f^2+\frac{1}{2}(\mu\mp\nu)^2}{2\mu f-p}=\frac{4r\nu^2+(\mu\mp\nu)^2\nu^2-q^2}{2\nu(\pm q\mu-p\nu)}; \quad (d)$$

and the root of the square is

$$\pm k=\mu\cos^2\tfrac{1}{2}\text{A}\pm\nu\sin^2\tfrac{1}{2}\text{A}\pm\frac{q}{2\nu}\sin\text{A}: \quad . \quad . \quad (e)$$

These equations give, in general, four values of A and k.

Case 2. When, only, $m=\mu^2$, a square number; then we may take, in equation (a),

$$d^2=\mu^2 \text{ and } p-2df=0; \qquad \text{or} \quad d=\mu, f=\frac{p}{2\mu};$$

and, after dividing by sin $^2\frac{1}{2}$A, it becomes, after some reduction,

$$\left(\mu^2+n+4r-\frac{p^2}{\mu^2}-2\mu e\right)\cos^2\tfrac{1}{2}\text{A}+(n-e^2)\sin^2\tfrac{1}{2}\text{A}+\left(q-\frac{pe}{\mu}\right)\sin\text{A}=0;$$

so that if we make

$$\mu^2+n+4r-\frac{p^2}{\mu^2}-2\mu e=0,$$

$$\text{or}\qquad e=\frac{\mu^4+(n+4r)\mu^2-p^2}{2\mu^3},\ \text{then}$$

$$\cot\tfrac{1}{2}\text{A}=\frac{\mu(e^2-n)}{(2q\mu-pe)}=\frac{\{\mu^4+(n+4r)\mu^2-p^2\}^2-4n\mu^6}{8q\mu^6-4p\mu^2\{\mu^4+(n+4r)\mu^2-p^2\}};\quad (f)$$

and the root of the square is

$$\pm k=\mu\cos^2\tfrac{1}{2}\text{A}+\frac{\mu^4+(n+4r)\mu^2-p^2}{2\mu^3}\sin^2\tfrac{1}{2}\text{A}+\frac{p}{2\mu}\sin\text{A}.\quad (g)$$

Case 3. When, only, $n=\nu^2$, a square number. This case can be deduced from the preceding one, by writing $90°-\frac{1}{2}$A instead of $\frac{1}{2}$A, and transposing the letters. So that, instead of (f) and (g), we have

$$\tan\tfrac{1}{2}\text{A}=\frac{\{\nu^4+(m+4r)\nu^2-q^2\}^2-4m\nu^6}{8p\nu^6-4q\nu^2\{\nu^4+(m+4r)\nu^2-q^2\}},\quad (h)$$

$$\pm k=\nu\sin^2\tfrac{1}{2}\text{A}+\frac{\nu^4+(m+4r)\nu^2-q^2}{2\nu^3}\cos^2\tfrac{1}{2}\text{A}+\frac{q}{2\nu}\sin\text{A}.\quad (i)$$

Several particular cases are deserving of notice, as occurring frequently.

Case 4. When $m=n=\mu^2$, or the equation is

$$k^2=\mu^2+(p\cos^2\tfrac{1}{2}\text{A}+q\sin^2\tfrac{1}{2}\text{A})\sin\text{A}+r\sin^2\text{A}.$$

Equations (b), (c), (d), (e) become for this case

$$\tan \tfrac{1}{2}A = \frac{4r\mu^2 - p^2}{2\mu^2(p-q)}, \qquad \text{or} = \frac{p^2 - 4\mu^2(r+\mu^2)}{2\mu^2(p+q)}, \quad . \quad . \quad (b_1)$$

$$\pm k = \mu + \frac{p}{2\mu}\sin A, \qquad \text{or} = \mu\cos A + \frac{p}{2\mu}\sin A; \quad . \quad (c_1)$$

$$\cot \tfrac{1}{2}A = \frac{4r\mu^2 - q^2}{2\mu^2(q-p)}, \qquad \text{or} = \frac{q^2 - 4\mu^2(r+\mu^2)}{2\mu^2(p+q)}, \quad . \quad (d_1)$$

$$\pm k = \mu + \frac{q}{2\mu}\sin A, \qquad \text{or} = \mu\cos A - \frac{q}{2\mu}\sin A. \quad . \quad (e_1)$$

Case 5. When $m = n = \mu^2$ and $q = -p$, or the equation is

$$k^2 = \mu^2 + p\sin A\cos A + r\sin^2 A.$$

Then the equations (b_1), (c_1), (d_1), (e_1) give the solutions

$$A = 180^\circ \text{ or } 0^\circ; \qquad k = \pm\mu.$$

$$\tan \tfrac{1}{2}A = \frac{4r\mu^2 - p^2}{4p\mu^2}, \qquad \cot \tfrac{1}{2}A = \frac{p^2 - 4r\mu^2}{4p\mu^2}, \quad . \quad (b_2)$$

$$\pm k = \mu + \frac{p}{2\mu}\sin A, \qquad \pm k = \mu - \frac{p}{2\mu}\sin A; \quad (c_2)$$

the two values of A differing by 180°.

Case 6. When $m = n = \mu^2$ and $q = p$, or the equation is

$$k^2 = \mu^2 + p\sin A + r\sin^2 A.$$

Then equations (b_1), (c_1), (d_1), (e_1) give the solutions

$$A = 180^\circ \text{ or } 0^\circ; \qquad k = \pm\mu.$$

$$\tan \tfrac{1}{2}A = \frac{p^2 - 4\mu^2(r+\mu^2)}{4p\mu^2}, \qquad \cot \tfrac{1}{2}A = \frac{p^2 - 4\mu^2(r+\mu^2)}{4p\mu^2}; \quad (b_3)$$

$$\pm k = \mu\cos A + \frac{p}{2\mu}\sin A, \qquad \pm k = \mu\cos A - \frac{p}{2\mu}\sin A; \quad (c_3)$$

the values of A being supplementary.

Problem XIV. To solve the equations

$$\frac{xyz}{x+y+z} = s^2, \qquad s^2 + y^2 = b^2,$$

$$s^2 + z^2 = a^2, \qquad s^2 + x^2 = c^2.$$

Cunliffe, Math. Companion, No. 27, p. 349.

Solution. These equations may be written

$$\frac{xyz}{x+y+z} = s^2 = a^2 - z^2 = b^2 - y^2 = c^2 - x^2,$$

and are the same as those of Problem XII., having

$$\mathbf{f}(z, y, x) = \frac{xyz}{x+y+z},$$

and using the complements of the angles A, B, C of that problem instead of the angles themselves, we have

$$z = s \tan \text{A}, \qquad y = s \tan \text{B}, \qquad x = s \tan \text{C};$$

and equation (a) becomes, after dividing it by s^2,

$$\frac{\tan \text{A} \tan \text{B} \tan \text{C}}{\tan \text{A} + \tan \text{B} + \tan \text{C}} = 1,$$

$$\tan \text{A} \tan \text{B} \tan \text{C} = \tan \text{A} + \tan \text{B} + \tan \text{C};$$

and therefore, as is well known,

$$\text{A} + \text{B} + \text{C} = 180°,$$

and, by equation (14), Chapter I.,

$$\text{A} = \varphi\left(\frac{m}{n}\right), \qquad \text{B} = \varphi\left(\frac{p}{q}\right), \qquad \text{C} = \varphi\left(\frac{mq+np}{mp-nq}\right).$$

Example. Take $m = 2$, $n = 1$, $p = 3$, $q = 2$, $s = \frac{165}{4}$; then

$$z = 55, \qquad y = 99, \qquad x = 70.$$

Problem XV. To find three numbers such that the square of each of them added to the sum of the products of every two of them may be square numbers.

Wright, Math. Companion, No. 17.

Solution. The equations to be solved are

$$z^2 + xy + xz + yz = a^2,$$

$$y^2 + xy + xz + yz = b^2,$$

$$x^2 + xy + xz + yz = c^2;$$

which are the same as

$$xy + xz + yz = a^2 - z^2 = b^2 - y^2 = c^2 - x^2;$$

and therefore by Problem XI., we may take

$$y = a \tan \text{A} - z \sec \text{A}, \qquad x = a \tan \text{B} - z \sec \text{B},$$

and equation (a) is

$$xy + xz + yz = a^2 - z^2;$$

or, by substitution,

$$\{2z \sin \tfrac{1}{2}\text{A} \sin \tfrac{1}{2}\text{B} - a \sin \tfrac{1}{2}(\text{A} + \text{B})\}^2 = a^2 \cos^2 \tfrac{1}{2}(\text{A} + \text{B}),$$

$$2z \sin \tfrac{1}{2}\text{A} \sin \tfrac{1}{2}\text{B} - a \sin \tfrac{1}{2}(\text{A} + \text{B}) \quad = \pm\, a \cos \tfrac{1}{2}(\text{A} + \text{B}).$$

Using the lower sign, and supposing each of the angles A and B to be less than 90°,

$$z = a - \tfrac{1}{2}a(\cot \tfrac{1}{2}\text{A} - 1)(\cot \tfrac{1}{2}\text{B} - 1).$$

The conditions necessary to render the three numbers positive, may be represented by

$$(\cot \tfrac{1}{2}\text{A} - 1)(\cot \tfrac{1}{2}\text{B} - 1) < 2 > 2(1 - \sin \text{A}) > 2(1 - \sin \text{B});$$

which show that $\cot \tfrac{1}{2}\text{A}$ must be between the limits

$$\cot \tfrac{1}{2}(90^\circ - \text{B}) \text{ and } \sin \text{B} + \cos \text{B},$$

while it must not be included within the limits

$$\frac{1+\sqrt{2-\cot^2\frac{1}{2}B}}{\cot\frac{1}{2}B-1} \quad\text{and}\quad \frac{1-\sqrt{2-\cot^2\frac{1}{2}B}}{\cot\frac{1}{2}B-1},$$

this latter condition being unnecessary when $\cot\frac{1}{2}B > \sqrt{2}$.

Example. If $\cot\frac{1}{2}B = 2$, it is only required to have

$$\cot\tfrac{1}{2}A < 3 > \tfrac{7}{5};$$

thus if $\cot\frac{1}{2}A = \frac{3}{2}$, we get $z = \frac{3}{4}a$, $y = \frac{9}{20}a$, $x = \frac{1}{12}a$; hence if $a = 60$, the numbers are 45, 27 and 5.

PROBLEM XVI. To find three square numbers, such that half the difference between the sum of every two and the third may be a rational square.

Cunliffe, Math. Companion, No. 14.

Solution. The equations to be solved are

$$\tfrac{1}{2}(x^2+y^2-z^2)=a^2,$$

$$\tfrac{1}{2}(x^2-y^2+z^2)=b^2,$$

$$\tfrac{1}{2}(-x^2+y^2+z^2)=c^2;$$

which may be written thus

$$\tfrac{1}{2}(z^2+y^2+x^2)=a^2+z^2=b^2+y^2=c^2+x^2;$$

and therefore we may take, as in Prob. IX.,

$$y=a\sin A+z\cos A, \qquad x=a\sin B+z\cos B,$$

and equation (a) becomes

$$\tfrac{1}{2}(z^2+y^2+x^2)=a^2+z^2,$$

or, $a^2(\cos^2A+\cos^2B)-az(\sin 2A+\sin 2B)+z^2(1-\cos^2A-\cos^2B)=0.$

The simplest solution of this equation is

$$B = 90^\circ - A,$$

$$a = 2z \sin 2A;$$

so that, by substitution,

$$y = z(2 \cos A - \cos 3A),$$

$$x = z(2 \sin A + \sin 3A).$$

Example. If $A = \varphi(2)$, $3A = \varphi(\frac{2}{11})$, Chap. I., equation (9), and if we take $z = 125$, the roots will be

244, 267, 125.

PROBLEM XVII. To find four numbers, such that their sum, and the sum of every two of them, may be square numbers.

Solution. The equations will be the same as those of Prob. VII., with the additional one

$$v + x + y + z = s^2;$$

so that the eliminated equations are

$$s^2 = a^2 + f^2 = b^2 + e^2 = c^2 + d^2.$$

Then, by Prob. X.,

$$a = s \cos A, \qquad b = s \cos B, \qquad c = s \cos C;$$

$$f = s \sin A, \qquad e = s \sin B, \qquad d = s \sin C.$$

So that

$$2v = s^2 (\cos^2 A + \cos^2 B - \sin^2 C),$$

$$2x = s^2 (\cos^2 A - \cos^2 B + \sin^2 C),$$

$$2y = s^2 (-\cos^2 A + \cos^2 B + \sin^2 C),$$

$$2z = s^2 (\sin^2 A - \cos^2 B + \cos^2 C).$$

These values include all the solutions of the question. The following cases may be noted:

1° Take $C = A + B$, then

$$v = s^2 \cos A \cos B \cos (A + B),$$

$$x = s^2 \cos A \sin B \sin (A + B),$$

$$y = s^2 \sin A \cos B \sin (A + B),$$

$$z = -s^2 \sin A \sin B \cos (A + B).$$

2°. Take $C = 90° - (A + B)$, then

$$2v = s^2\{1 + 2 \sin A \sin B \cos (A + B)\},$$

$$2x = s^2\{1 - 2 \sin A \cos B \sin (A + B)\},$$

$$2y = s^2\{1 - 2 \cos A \sin B \sin (A + B)\},$$

$$2z = s^2\{1 - 2 \cos A \cos B \cos (A + B)\}.$$

3°. Take $B = A + B$, $C = A - B$; then

$$2v = s^2(\cos {}^2A + \cos 2A \cos 2B),$$

$$2x = s^2(\cos {}^2A - \cos 2A \cos 2B),$$

$$2y = s^2(\sin {}^2A - \sin 2A \sin 2B),$$

$$2z = s^2(\sin {}^2A + \sin 2A \sin 2B).$$

The first combination, although the neatest analytical forms, must have one of the numerical results negative; but the second and third include positive results within limits easily assigned.

Example 1. Let $\text{A} = \varphi(\frac{3}{2})$, $\text{B} = \varphi(2)$, $\text{A} + \text{B} = \varphi(\frac{4}{7})$, $\text{A} - \text{B} = \varphi(8)$; then taking $s = 65$, the second set gives

$$v = 528\tfrac{1}{2}, \quad x = 96\tfrac{1}{2}, \quad y = 992\tfrac{1}{2}, \quad z = 2607\tfrac{1}{2};$$

and these may be multiplied by 4, to make them integral.

Example 2. Let $\text{A} = \varphi(2)$, $\text{B} = \varphi(6)$, $2\text{A} = \varphi(\frac{3}{4})$, $2\text{B} = \varphi(\frac{35}{12})$; then taking $s = 185$, the third set gives

$$v = 2377, \quad x = 9944, \quad y = 872, \quad z = 21032.$$

Problem XVIII. To find four numbers, such that their sum, and the sum of every two of them, may be square numbers; and also twice the sum of the first three may be a square number.

Math. Companion, No. 7, Quest. 15.

Solution. The equations will be the same as those of Prob. VII., with the two additional ones

$$2(v + x + y) = g^2, \qquad v + x + y + z = s^2;$$

and the eliminated equations will be

$$s^2 = a^2 + f^2 = b^2 + e^2 = c^2 + d^2,$$

$$g^2 = a^2 + b^2 + d^2.$$

Then, by Prob. X., we may take

$$a = s \sin \text{A}, \qquad b = s \sin \text{B}, \qquad c = s \sin \text{C};$$

$$f = s \cos \text{A}, \qquad e = s \cos \text{B}, \qquad d = s \cos \text{C};$$

and the last equation becomes, by substitution,

$$g^2 = s^2(\sin^2 \text{A} + \sin^2 \text{B} + \cos^2 \text{C}).$$

Take, as one of the simplest solutions of this equation,

$$\text{C} = \text{A} - \text{B},$$

then it will be

$$g^2 = s^2(1 + \sin 2\text{B} \sin \text{A} \cos \text{A} + 2 \sin^2\text{B} \sin^2\text{A}),$$

which is the equation of Prob. XIII., Case 5, so that

$$g = s(1 \pm \tfrac{1}{2} \sin 2\text{B} \sin \text{A}),$$

$$\tan \tfrac{1}{2}\text{A} = \tfrac{1}{2} \tan \text{B} (1 + \sin^2\text{B}),$$

or $$\cot \tfrac{1}{2}\text{A} = -\tfrac{1}{2} \tan \text{B} (1 + \sin^2\text{B}).$$

Taking the first result, and putting

$$s = t(4 - \sin^2 2\text{B} + \sin^2\text{B} \cos^4\text{B}),$$

we shall have

$$a = 2t \sin 2\text{B} (1 + \sin^2\text{B}),$$

$$b = t \sin \text{B} (4 - \sin^2 2\text{B} + \sin^2\text{B} \cos^4\text{B}),$$

$$c = t \sin^3\text{B} (4 + \cos^4\text{B}),$$

$$d = t \cos \text{B} (4 - \sin^2\text{B} \cos^4\text{B}),$$

$$e = t \cos \text{B} (4 - \sin^2 2\text{B} + \sin^2\text{B} \cos^4\text{B}),$$

$$f = t (4 \cos^2\text{B} - 4 \sin^4\text{B} - \sin^2\text{B} \cos^4\text{B}).$$

Example. Take $\text{B} = \varphi\,(3)$, $2\,\text{B} = \varphi(\tfrac{4}{3})$ and $t = \tfrac{1}{2}.\,5^7$; then

$$a = 102000, \quad b = 75606, \quad c = 37206,$$

$$d = 120392, \quad e = 100808, \quad f = 63790;$$

and, by the formulas of Prob. VII,

$$v = 813016786, \qquad x = 9590983214,$$

$$y = 4903250450, \qquad z = 571269650.$$

Problem XIX. To solve the equations.

$$x^2+y^2+e\,y\,z=a^2,$$
$$x^2+z^2+e\,y\,z=b^2,$$
$$y^2+z^2+e\,y\,z=c^2\,.$$

e being any given number

Math. Comp. No. 7, Quest. 14.

Solution. The given equations may be written

$$x^2+y^2+z^2+e\,y\,z=a^2+z^2=b^2+y^2=c^2+x^2\,;$$

which are those of Problem IX. Taking then

$$y=a\sin\text{A}+z\cos\text{A},\quad x=a\sin\text{B}+z\cos\text{B}\,;$$

equation (a) becomes, by substitution,

$$z^2\,(\cos^2\text{A}+\cos^2\text{B}+e\cos\text{A})+az\,(\sin 2\text{A}+\sin 2\text{B}+e\sin\text{A})$$
$$=a^2\,(1-\sin^2\text{A}-\sin^2\text{B}).$$

Let $\text{B}=90°-\text{A}$, then this becomes

$$z\,(1+e\cos\text{A})+a\,(2\sin 2\text{A}+e\sin\text{A})=0\,;$$

so that we may take

$$z=t\,(e\sin\text{A}+2\sin 2\text{A}),\qquad a=-t\,(1+e\cos\text{A})\,;$$

and then

$$y=t\sin 3\text{A}\quad,\qquad x=\pm t\,(e\cos 2\text{A}+\cos 3\text{A}).$$

Example 1. Let $e=1$, and take $\text{A}=\varphi\,(3)$, $2\text{A}=\varphi\,(\frac{4}{3})$, $3\text{A}=\varphi\,(\frac{9}{13})$ and $t=\frac{125}{9}$; then

$$z=35,\quad y=13,\quad x=1.$$

Example 2. Let $e=-1$, and take $\text{A}=\varphi(2)$, $2\text{A}=\varphi(\frac{3}{4})$, $3\text{A}=\varphi(\frac{2}{11})$ and $t=\frac{125}{2}$; then

$$z=70,\quad y=22,\quad x=41.$$

PROBLEM XX. To solve the equations

$$x^2+y^2+xy+xz+yz=a^2,$$

$$x^2+z^2+xy+xz+yz=b^2,$$

$$y^2+z^2+xy+xz+yz=c^2.$$

Math. Companion, No. 8, Quest. 26.

Solution. The given equations may be written

$$x^2+y^2+z^2+xy+xz+yz=a^2+z^2=b^2+y^2=c^2+x^2,$$

which are those of Prob. IX. Taking, then,

$$y=a\sin A-z\cos A,\quad x=a\sin B-z\cos B;$$

equation (a) becomes, by substitution,

$$a^2(\cos^2 B-\sin A\sin B-\sin^2 A)$$

$$+az\{\sin 2A+\sin 2B+\sin(A+B)-\sin A-\sin B\}$$

$$=z^2(\cos^2 A+\cos^2 B+\cos A\cos B-\cos A-\cos B).$$

Now put

$$2a(\cos^2 B-\sin A\sin B-\sin^2 A)$$

$$+z\{\sin 2A+\sin 2B+\sin(A+B)-\sin A-\sin B\}=\pm kz;$$

then we shall find,

$$k^2=16\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}B\cos^2\tfrac{1}{2}(A-B)+4\cos A\cos B\cos(A-B)$$

$$=(2\cos A\cos B+\sin A\sin B)^2$$

$$+16\sin^2\tfrac{1}{2}A\sin^2\tfrac{1}{2}B\{\cos^2\tfrac{1}{2}(A-B)-\cos^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}B\}.$$

Hence, if we take

$$\cos \tfrac{1}{2} (A - B) = - \cos \tfrac{1}{2}A \cos \tfrac{1}{2}B,$$

or $$\cot \tfrac{1}{2}A = - \tfrac{1}{2} \tan \tfrac{1}{2}B.$$

we shall have

$$k = 2 \cos A \cos B + \sin A \sin B,$$

$$a(2 + 3\cos B + 3\cos^2 B) = z\{\mp (5 + 3\cos B) - \sin B(1 + 3\cos B)\}.$$

Then taking

$$a = t\{5 + 3 \cos B - \sin B (1+3 \cos B)\},$$

we shall get

$$z = t (2 + 3 \cos B + 3 \cos^2 B,$$
$$y = t (2 - 4 \sin B + 5 \cos B + \cos^2 B),$$
$$x = t \{(5 + 3 \cos B) (\sin B - \cos B) - \sin^2 B).$$

To render all these numbers positive, it will be found necessary to take

$$\text{either, } \cot \tfrac{1}{2}B > 1, 412 < 2, 24;$$

$$\text{or else, } \cot \tfrac{1}{2}B > -, 354 < -, 225.$$

Example 1. If $B = \varphi(2)$ and $t = \frac{25}{2}$, we get

$$x = 9, \quad y = 27, \quad z = 61.$$

Example 2. If $B = \varphi(\frac{3}{2})$ and $t = \frac{13}{32}$, we get

$$x = 13, \quad y = 2, \quad z = 19.$$

Problem XXI. To solve the three equations

$$x^2 + y^2 + 2z^2 = a^2,$$

$$x^2 + 2y^2 + z^2 = b^2,$$

$$2x^2 + y^2 + z^2 = c^2.$$

Math. Repository, No. 4, Quest. 95.

Solution. The given equations may be written

$$x^2 + y^2 + z^2 = a^2 - z^2 = b^2 - y^2 = c^2 - x^2,$$

which are those of Prob. XI. Then, taking

$$y = a \tan A + z \sec A, \quad x = a \tan B + z \sec B;$$

and substituting in equation (a), it becomes

$$a^2 (\tan^2 A + \tan^2 B) + 2az (\tan A \sec A + \tan B \sec B)$$
$$+ z^2 (2 + \sec^2 A + \sec^2 B) = a^2.$$

Now, put

$$a(1 - \tan^2 A + \tan^2 B) - z(\tan A \sec A + \tan B \sec B) = \pm kz \sec A \sec B;$$

and we shall find

$$k^2 = 6 \cos^2 A \cos^2 B + 2 \sin A \sin B - 2.$$

Assume $\quad p^2 = 6 \cos^2 B - 2 = 1 + 3 \cos 2B, \ldots (a)$

and it becomes

$$k^2 = p^2 \cos^2 A + 2 \sin B \sin A - 2 \sin^2 A;$$

which is the equation of Prob. XIII. Case 6; so that

$$k = p \cos A + \frac{\sin B \sin A}{p}$$

$$\tan \tfrac{1}{2} A = \frac{2p^2 \mp \sin^2 B}{2p^2 \sin B}$$

With regard to equation (a), no general solution of it can be given; but, if B' be a particular value, such as $B' = \varphi(2)$, which renders

$$p'^2 = 1 + 3 \cos 2B',$$

by putting $$B = B' + \theta,$$ it becomes

$$p^2 = p'^2 - 3 \sin 2B' \sin 2\theta - 6 \cos 2B' \sin^2 \theta;$$

and, by Prob. XIII, Case 5,

$$p = p' - \frac{3 \sin 2B'}{p'} . \sin \theta,$$

$$\tan \tfrac{1}{2}\theta = \frac{p'^4 + 8}{2p'^2 \sin 2B'};$$

and thus other solutions may be derived.

Taking the case $B = \varphi(2)$, $p = \frac{2}{5}$, we have $\tan \frac{1}{2}A = \frac{15}{4}$,

$$18929\, a = (\pm 21318 - 49300)\, z.$$

By using the upper sign we get the numbers

7, 7, 23

for x, y, z; and by using the lower ones

18719, 18929, 62609.

Problem XXII. To find three square numbers, such, that the sum of every two of them is a square number.

Solution. We have to solve the equations

$$x^2 + y^2 = a^2, \qquad x^2 + z^2 = b^2, \qquad y^2 + z^2 = c^2;$$

or, as they may be written,

$$x^2 + y^2 + z^2 = a^2 + z^2 = b^2 + y^2 = c^2 + x^2,$$

which are the equations of Prob. IX.; so we may take

$$y = a \sin A - z \cos A, \qquad x = a \sin B - z \cos B;$$

and equation (a) of that Problem, or the first of the given equations, becomes

$$z^2(\cos^2 A + \cos^2 B) - az(\sin 2A + \sin 2B) + a^2(\sin^2 A + \sin^2 B - 1) = 0.$$

The most simple solution of this equation is

$$B = 90^\circ - A, \qquad z = 2a \sin 2A;$$

so that the roots of the three squares are

$$z = 2a \sin 2A, \qquad y = -a \cos 3A, \qquad x = a \sin 3A.$$

Example. By taking $A = \varphi(2)$, $2A = \varphi(\frac{3}{4})$, $3A = \varphi(\frac{2}{11})$, and $a = 125$, we find the three roots

240, 117, 44.

Cor. Let h be the hypothenuse of a right-angled triangle, a, b the two sides, and A the angle opposite a, so that

$$a = h \sin A, \qquad b = h \cos A.$$

Multiply the three preceding expressions for x, y, z, by $\frac{h^3}{a}$, and they may be written

$$4h^3 \sin A \cos A, \ h^3 \cos A(4 \sin^2 A - 1), \ h^3 \sin A(4 \cos^2 A - 1);$$

or, $\qquad 4abh, \qquad b(4a^2 - h^2), \qquad a(4b^2 - h^2).$

These are the expressions for the roots of the squares given by Dr. Saunderson, who first found the numbers 240, 117, 44.

PROBLEM XXIII. To find three square numbers, such, that the difference of every two of them is a square number.

Solution. Here the equations are

$$x^2 - y^2 = a^2, \qquad x^2 - z^2 = b^2, \qquad y^2 - z^2 = c^2.$$

The two first give by Prob. I., Cor.,

$$y = x \cos \text{A}, \qquad z = x \cos \text{B};$$

and the third becomes

$$c^2 = x^2(\cos^2 \text{A} - \cos^2 \text{B}) = x^2 \sin(\text{B} + \text{A}) \sin(\text{B} - \text{A}),$$

an equation which admits of a vast variety of solutions, of which we will notice two or three.

Take $\text{B} = \text{A} + \text{C}$, and it becomes

$$c^2 = x^2 \sin \text{C} \sin(2\text{A} + \text{C})$$
$$= x^2 \sin^2 \text{C} \,(1 + \cot \text{C} \sin 2\text{A} - 2 \sin^2 \text{A}).$$

Then, by Prob. XIII., Case 5,

$$c = x \sin \text{C}\,(1 - \cot \text{C} \sin \text{A},)$$

$$\cot \tfrac{1}{2}\text{A} = \frac{2 + \cot^2 \text{C}}{2 \cot \text{C}} = \frac{1 + \sin^2 \text{C}}{\sin 2\text{C}}.$$

Now take $\qquad x = t\{(1 + \sin^2 \text{C})^2 + \sin^2 2\text{C}\};$

then $\qquad y = t\{(1 + \sin^2 \text{C})^2 - \sin^2 2\text{C}\},$

$$z = t \cos \text{C}(4 \sin^2 \text{C} - \cos^4 \text{C}).$$

Example. If $\text{C} = \varphi(3)$, $2\text{C} = \varphi(\frac{4}{3})$, $t = \frac{1}{4} \cdot 5^5$, the numbers are

$$2165, \qquad 725, \qquad 644.$$

Again, since $\text{A} = \text{B} - \text{C}$, the equation is

$$c^2 = x^2 \sin \text{C} \sin(2\text{B} - \text{C})$$
$$= 2x^2 \sin \text{C} \sin(\text{B} - \tfrac{1}{2}\text{C}) \cos(\text{B} - \tfrac{1}{2}\text{C})$$
$$= x^2 \sin^2 \text{C}\,(\sin \text{B} \cot \tfrac{1}{2}\text{C} - \cos \text{B})(\cos \text{B} + \sin \text{B} \tan \tfrac{1}{2}\text{C}).$$

Now put $$\sin B \cot \tfrac{1}{2}C - \cos^2 \tfrac{1}{2}B = 0,$$

or $$2 \cot \tfrac{1}{2}C = \cot \tfrac{1}{2}B;$$

then this becomes

$$\begin{aligned} c^2 &= x^2 \sin^2 C \sin^2 \tfrac{1}{2}B (\cos B + \sin B \tan \tfrac{1}{2}C) \\ &= x^2 \sin^2 C \sin^4 \tfrac{1}{2}B (\cot^2 \tfrac{1}{2}B + 2 \cot \tfrac{1}{2}B \tan \tfrac{1}{2}C - 1) \\ &= x^2 \sin^2 C \sin^4 \tfrac{1}{2}B \,.\, k^2, \end{aligned}$$

in which, by substituting the value of C,

$$k^2 = \cot^2 \tfrac{1}{2}B + 3, \qquad \text{or} \quad k^2 - \cot^2 \tfrac{1}{2}B = 3,$$

whence, by Problem III.,

$$k = \frac{2 + \cos D}{\sin D}, \quad \cot \tfrac{1}{2}B = \frac{1 + 2\cos D}{\sin D}, \quad \cot \tfrac{1}{2}C = \frac{1 + 2 \cos D}{2 \sin D};$$

and we get the roots of the numbers

$$\begin{aligned} x &= t \{(1 + 2 \cos D)^4 + 5 \sin^2 D (1 + 2 \cos D)^2 + 4 \sin^4 D\}, \\ y &= t \{(1 + 2 \cos D)^4 + 3 \sin^2 D (1 + 2 \cos D)^2 + 4 \sin^4 D\}, \\ z &= t \{(1 + 2 \cos D)^4 + 3 \sin^2 D (1 + 2 \cos D)^2 - 4 \sin^2 D\}. \end{aligned}$$

Example. If $D = \varphi(\tfrac{1}{2})$, $t = 5^4$, the roots are

1105, 1073, 975.

Once more, in the equation

$$\begin{aligned} c^2 &= x^2 \sin^2 C (\sin B \cot \tfrac{1}{2}C - \cos B)(\cos B + \sin B \tan \tfrac{1}{2}C) \\ &= \tfrac{1}{4}x^2 \sin^2 B \sin^2 C (\tan^2 \tfrac{1}{2}B + 2 \tan \tfrac{1}{2}B \cot \tfrac{1}{2}C - 1) \\ &\qquad \times (\cot^2 \tfrac{1}{2}B + 2 \cot \tfrac{1}{2}B \tan \tfrac{1}{2}C - 1), \end{aligned}$$

put $$\cot^2 \tfrac{1}{2}B + 2 \cot \tfrac{1}{2}B \tan \tfrac{1}{2}C - 1 = (\cot \tfrac{1}{2}B - \cot \tfrac{1}{2}D)^2,$$

we get

$$\tan \tfrac{1}{2}B = 2 \sin^2 \tfrac{1}{2}D (\cot \tfrac{1}{2}D + \tan \tfrac{1}{2}C) = \sin D + 2 \sin^2 \tfrac{1}{2}D \tan \tfrac{1}{2}C,$$

and, by substitution,

$$c^2 = \tfrac{1}{4}x^2 \sin^2 B(\cot \tfrac{1}{2}B - \cot \tfrac{1}{2}D)^2 \,.\, k^2,$$

in which,

$$k^2 = \sin^2 C (\tan^2 \tfrac{1}{2}B + 2 \tan \tfrac{1}{2}B \cot \tfrac{1}{2}C - 1)$$

$$= 16 \sin^4 \tfrac{1}{2}D \sin^2 \tfrac{1}{2}C + 4 \sin D \sin C(\cos^2 \tfrac{1}{2}C + 2 \sin^2 \tfrac{1}{2}D \sin^2 \tfrac{1}{2}C)$$

$$- \cos 2D \sin^2 C.$$

If the formulas of Problem XIII., Case 1, be applied to this equation, there will result precisely the last solution; but if we apply the formulas of Case 3, we get

$$k = 4 \sin^2 \tfrac{1}{2}D \sin^2 \tfrac{1}{2}C + \frac{1 - 2 \cos D}{2 \sin^2 \tfrac{1}{2}D} \cos^2 \tfrac{1}{2}C + \sin D \sin C,$$

$$\tan \tfrac{1}{2}C = \frac{(1 - 2 \cos D)^2}{8 \sin D \sin^2 \tfrac{1}{2}D};$$

then $\tan \tfrac{1}{2}B = \dfrac{5 - 4 \cos D}{4 \sin D}$, $\tan \tfrac{1}{2}A = \dfrac{4 \sin D(4 - 5 \cos D)}{21 - 40 \cos D + 20 \cos^2 D}$;

and the roots of the numbers are

$$x = t(41 - 40 \cos D)(17 - 40 \cos D + 24 \cos^2 D),$$

$$y = t\{(21 - 40 \cos D + 20 \cos^2 D)^2 - 16 \sin^2 D(4 - 5 \cos D)^2\},$$

$$z = t(9 - 40 \cos D + 32 \cos^2 D)(17 - 40 \cos D + 24 \cos^2 D).$$

Example. If $D = 90°$, the roots are

697, 185, 153.

PROBLEM XXIV. To find three square numbers, such that the differences between the sum of every two of them and the third may be square numbers.

Solution. The equations to be solved are

$$x^2+y^2-z^2=a^2,\quad x^2-y^2+z^2=b^2,\quad -x^2+y^2+z^2=c^2;$$

or, by addition,

$$2x^2=a^2+b^2,\qquad 2y^2=a^2+c^2,\qquad 2z^2=b^2+c^2.$$

The two first, by Prob. II., Cor. 1, give

$$a=x\sqrt{2}\sin(45^\circ+\text{A}),\qquad b=x\sqrt{2}\cos(45^\circ+\text{A});$$

$$a=y\sqrt{2}\sin(45^\circ+\text{B}),\qquad c=y\sqrt{2}\cos(45^\circ+\text{B}).$$

To make the two values of a equal, we may take

$$x=t\sqrt{2}\sin(45^\circ+\text{B})\qquad y=t\sqrt{2}\sin(45^\circ+\text{A});$$

then $b+c=2t\cos(\text{A}+\text{B}),\qquad b-c=2t\sin(\text{B}-\text{A});$

$$z^2=\tfrac{1}{2}(b^2+c^2)=\tfrac{1}{4}\{(b+c)^2+(b-c)^2\}$$

$$=t^2\{\cos^2(\text{A}+\text{B})+\sin^2(\text{B}-\text{A})\}$$

$$=t^2(1-\sin 2\text{B}\sin 2\text{A}).$$

So that, by Prob. XIII., Case 5, we have

$$z=t(1-\sin 2\text{B}\sin\text{A}),$$

$$\tan\tfrac{1}{2}\text{A}=\tfrac{1}{2}\sin 2\text{B}.$$

and if we put $\qquad t=t'(4+\sin^2 2\text{B}),\qquad$ we shall have

$$x=t'(\cos\text{B}+\sin\text{B})(4+\sin^2 2\text{B}),$$

$$y=t'(4+4\sin 2\text{B}-\sin^2 2\text{B}),$$

$$z=t'(4-3\sin^2 2\text{B}).$$

Example. If $B = \varphi(\frac{1}{2})$, and $t' = \frac{1}{4} \cdot 5^5$, the roots will be

769, 595, 965.

Another Solution. The first equation gives, as before,

$$a = x\sqrt{2} \sin (45^\circ + A), \qquad b = x\sqrt{2} \cos (45^\circ + A);$$

but the first and second equations give

$$a^2 = 2x^2 - b^2 = 2y^2 - c^2, \quad \text{or} \quad b^2 - c^2 = 2(x^2 - y^2).$$

Then, by Prob. III., Cor.,

$$2b \sin B = x(3 + \cos B) + y(1 + 3 \cos B),$$

$$2c \sin B = x(1 + 3 \cos B) + y(3 + \cos B);$$

or, by comparing the values of b, we may take

$$x = t(1 + 3 \cos B),$$

$$y = t\{2\sqrt{2} \sin B \cos (45^\circ + A) - 3 - \cos B\},$$

$$b = t\sqrt{2}(1 + 3 \cos B) \cos (45^\circ + A),$$

$$c = t\{\sqrt{2}(3 + \cos B) \cos (45^\circ + A) - 4 \sin B\}.$$

Hence the third equation becomes

$$z^2 = t^2 \{\mu^2 \cos^2 \tfrac{1}{2} A + \nu^2 \sin^2 \tfrac{1}{2} A + (p \cos^2 \tfrac{1}{2} A + q \sin^2 \tfrac{1}{2} A) \sin A\},$$

where $\mu = 3 + \cos B - 2 \sin B, \quad \nu = 3 + \cos B + 2 \sin B,$

$$p = 4 \sin B (3 + \cos B) - 2(3 + \cos B)^2 + 4 \sin^2 B,$$

$$q = 4 \sin B (3 + \cos B) + 2(3 + \cos B)^2 - 4 \sin^2 B;$$

and using the upper signs in equations (d), (e) of Prob. XIII.,

$$z = t\{\mu \cos^2 \tfrac{1}{2} A + \nu \sin^2 \tfrac{1}{2} A + \frac{q}{2\nu} \sin A\},$$

$$\cot \tfrac{1}{2} A = -\frac{3 + \cos B + 4 \sin B}{2(3 + \cos B + 2 \sin B)}.$$

If we put $t=t'\{5(3+\cos B)^2+24\sin B(3+\cos B)+32\sin^2 B\}$,

we get

$$x=t'(3\cos B+1)\{5(3+\cos B)^2+24\sin B(3+\cos B)+32\sin^2 B\},$$

$$y=t'\{5(3+\cos B)^3+22\sin B(3+\cos B)^2-64\sin^3 B\},$$

$$z=t'\{(3+\cos B+2\sin B)^3+4\sin^2 B(15+5\cos B+14\sin B)\}.$$

Example. If $B=90°$, $t'=1$, we have the roots

$$149, \quad 269, \quad 241.$$

PROBLEM XXV. To find three square numbers, such that the difference between twice the sum of any two of them and the third, may be square numbers.

Solution. The equations to be solved are

$$2x^2+2y^2-z^2=a^2, \quad 2x^2+2z^2-y^2=b^2, \quad 2y^2+2z^2-x^2=c^2.$$

The two first may be written

$$a^2+z^2=(x+y)^2+(x-y)^2, b^2+y^2=(x+z)^2+(x-z)^2;$$

and give, by Prob. II.,

$$z=(x+y)\cos A-(x-y)\sin A=x(\cos A-\sin A)+y(\cos A+\sin A),$$

$$y=(x+z)\cos B-(x-z)\sin B=x(\cos B-\sin B)+z(\cos B+\sin B).$$

Eliminating z between these equations, we have

$$y\{1-(\cos A+\sin A)(\cos B+\sin B)\}=x\{\cos B-\sin B+(\cos B+\sin B)(\cos A-\sin A)\};$$

whence we may take, to solve the first two equations,

$$x=t\{1-(\cos A+\sin A)(\cos B+\sin B)\},$$

$$y=t\{\cos B-\sin B+(\cos A-\sin A)(\cos B+\sin B)\},$$

$$z=t\{\cos A-\sin A+(\cos B-\sin B)(\cos A+\sin A)\}.$$

Then the third equation becomes

$$c^2 = 2y^2 + 2z^2 - x^2$$

$$= t^2\{\mu^2\cos^2\tfrac{1}{2}A + \nu^2\sin^2\tfrac{1}{2}A + (p\cos^2\tfrac{1}{2}A + q\sin^2\tfrac{1}{2}A)\sin A + r\sin^2 A\};$$

in which,

$$\mu = 1 + 3\cos B - \sin B, \quad \nu = 1 + \cos B - 3\sin B,$$

$$p = -2(1 - \cos B - \sin B + 5\sin 2B + 4\cos^2 B),$$

$$q = 2(1 + \cos B + \sin B + 5\sin 2B + 4\sin^2 B),$$

$$r = 8(\sin B - \cos B).$$

From the equations of Prob. XIII., Case 1, either

$$c = t\{\mu\cos^2\tfrac{1}{2}A \pm \nu\sin^2\tfrac{1}{2}A + \frac{p}{2\mu}\sin A\},$$

$$\text{I. } \tan\tfrac{1}{2}A = \frac{16 + 13\sin B - 8\cos B - 5\sin 2B - 2\sin^2 B}{(2 + \sin B + 2\cos B)(5 + 4\sin B - 4\cos B)},$$

$$\text{II. } \tan\tfrac{1}{2}A = \frac{(1 - \sin B)(5 - \cos B + 4\sin B)}{(\sin B + \cos B)(1 + 3\cos B - \sin B)};$$

as we use the upper or lower sign ; or

$$c = t\{\mu\cos^2\tfrac{1}{2}A \pm \nu\sin^2\tfrac{1}{2}A \pm \frac{q}{2\nu}\sin A\},$$

$$\text{III. } \cot\tfrac{1}{2}A = -\frac{16 - 13\cos B + 8\sin B - 5\sin 2B - 2\cos^2 B}{(2 - \cos B - 2\sin B)(5 + 4\sin B - 4\cos B)},$$

$$\text{IV. } \cot\tfrac{1}{2}A = \frac{(1 + \cos B)(5 + \sin B - 4\cos B)}{(\sin B + \cos B)(1 + \cos B - 3\sin B)}.$$

Example 1. If $B = 0$, (I.) gives $\tan\frac{1}{2}A = 2$, and if we take $t = \frac{5}{3}$, we get for the roots 1, 3, 4.

Example 2. If $B = 90°$, (IV.) gives $\cot\frac{1}{2}A = -3$, and taking $t = \frac{5}{2}$, the roots are 1, 2, 3.

Example 3. If $B = \varphi(\frac{1}{2})$, (II.) gives $\tan \frac{1}{2}A = -5\frac{1}{2}$, and the values of x, y, z are 131, 158, 127; the corresponding values of a, b, c being 261, 204, 255.

Example 4. If $B = \varphi(2)$, the four formulas give

$$\cot \tfrac{1}{2}A = \frac{145}{37}, \quad \frac{35}{19}, \quad \frac{227}{29}, \quad \frac{34}{7};$$

and taking for t the four separate values

$$\frac{76085}{2}, \quad \frac{3965}{6}, \quad \frac{130925}{2}, \quad \frac{6025}{12},$$

the roots of the squares will be as in the following table:

	$x =$	$y =$	$z =$	$a =$	$b =$	$c =$
I.	31513	36508	30579	60965	50234	59521;
II.	619	404	377	975	942	477;
III.	46282	52571	30939	94097	58607	72800;
IV.	134	823	607	1011	309	1440.

It is known that, if x, y, z represent the three sides of a plane triangle, $\frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c$ are the lengths of the lines drawn from the three angles of this triangle to bisect the opposite sides;—such will be the case with the numbers in Example 3, and the first three sets of numbers in Example 4.

Problem XXVI. To solve the equations

$$x^2 + axy + dy^2 = g^2,$$

$$x^2 + bxz + ez^2 = h^2,$$

$$y^2 + cyz + fz^2 = i^2;$$

a, b, c, d, e, f being given numbers.

Wallace, Math. Repository, Quest. 310.

Solution. By Prob. IV., the two first equations, are evidently satisfied by taking

$$\frac{y}{x}=\frac{\sin A+a\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A-d\sin^2\frac{1}{2}A}=\frac{a+2\cot\frac{1}{2}A}{\cot^2\frac{1}{2}A-d}=\alpha,$$

$$\frac{z}{x}=\frac{\sin B+b\sin^2\frac{1}{2}B}{\cos^2\frac{1}{2}B-e\sin^2\frac{1}{2}B}=\frac{b+2\cot\frac{1}{2}B}{\cot^2\frac{1}{2}B-e}=\beta;$$

so that $\qquad\frac{z}{y}=\frac{\beta}{\alpha},\qquad$ and the third becomes

$$\frac{i^2}{y^2}=1+c\,.\,\frac{z}{y}+f\,.\,\frac{z^2}{y^2}=1+c\,.\,\frac{\beta}{\alpha}+f\,.\,\frac{\beta^2}{\alpha^2}.$$

Writing in this the value of β, and putting

$$k^2y^2=\alpha^2i^2(\cos^2\tfrac{1}{2}B-e\sin^2\tfrac{1}{2}B)^2,$$

we shall find, after multiplying by $\alpha^2\ (\cos^2\frac{1}{2}B-e\sin^2\frac{1}{2}B)^2$,

$$k^2=\alpha^2(\cos^2\tfrac{1}{2}B-e\sin^2\tfrac{1}{2}B)^2+c\alpha(\cos^2\tfrac{1}{2}B-e\sin^2\tfrac{1}{2}B)(\sin B+b\sin^2\tfrac{1}{2}B)$$
$$+f(\sin B+b\sin^2\tfrac{1}{2}B)^2$$

$$=\alpha^2\cos^2\tfrac{1}{2}B+n\sin^2\tfrac{1}{2}B+(c\alpha\cos^2\tfrac{1}{2}B+q\sin^2\tfrac{1}{2}B)\sin B+r\sin^2B;$$

in which,

$$n=\alpha^2e^2-bce\alpha+b^2f,\qquad q=2bf-ce\alpha,$$

$$4r=-(e+1)^2\alpha^2+(e+1)bc\alpha+(4-b^2)f.$$

Then, by Prob. XIII., Case 2,

$$k=\alpha\cos^2\tfrac{1}{2}B-\frac{2e\alpha^2-bc\alpha+c^2-4f}{2\alpha}\sin^2\tfrac{1}{2}B+\tfrac{1}{2}c\sin B,$$

$$\cot\tfrac{1}{2}B=\frac{(c-b\alpha)^2+4e\alpha^2-4f}{4\alpha(c-b\alpha)},$$

$$\beta=\frac{8\alpha(c-b\alpha)\{(4e-b^2)\alpha^2-4f+c^2\}}{\{4e\alpha^2+4f-(c-b\alpha)^2\}^2-64ef\alpha^2}.$$

That this solution may give positive values for x, y, z, it is necessary, and it is sufficient, that α and β should be positive quantities; and, to assist in determining the limits within which this takes place, it may be well to observe that the denominator of the value of β is the product of the four factors

$$(2\sqrt{e}+b)\alpha+2\sqrt{f}-c, \qquad (2\sqrt{e}-b)\alpha+2\sqrt{f}+c,$$

$$(2\sqrt{e}+b)\alpha-2\sqrt{f}-c, \qquad (2\sqrt{e}-b)\alpha-2\sqrt{f}+c.$$

Example. Let the equations be

$$x^2+xy-2y^2=g^2,\ x^2-xz+3z^2=h^2,\ y^2+3yz+2z^2=i^2;$$

then $a=-b=1$, $c=3$, $d=-2$, $e=3$, $f=2$; so that

$$\frac{y}{x}=\alpha=\frac{2+\tan\frac{1}{2}\text{A}}{\cot\frac{1}{2}\text{A}+2\tan\frac{1}{2}\text{A}},$$

$$\frac{z}{x}=\beta=\frac{8\alpha(\alpha+3)(11\alpha^2+1)}{(11\alpha^2-6\alpha-1)^2-384\alpha^2}.$$

To make β positive, α must be excluded from the limits

$$2{,}3653 \quad \text{and} \quad {,}06964;$$

therefore $\cot\frac{1}{2}\text{A}$ must be excluded from within the limits

$$29{,}14 \quad \text{and} \quad -{,}42;$$

while, to make α positive, we must have $\cot\frac{1}{2}\text{A} > -\frac{1}{2}$. That is, we must either have

$$\cot\tfrac{1}{2}\text{A} > 29{,}14;\ \text{or else}\ \cot\tfrac{1}{2}\text{A} < -{,}42 > -{,}5.$$

If we take $\cot\frac{1}{2}\text{A}=31$, we shall find

$$x=146180939, \quad y=9563239, \quad z=1472557968.$$

Case. The third equation might be solved, like the two first, by taking

$$\frac{z}{y}=\frac{\beta}{\alpha}=\frac{\sin \text{C}+c\sin^2\frac{1}{2}\text{C}}{\cos^2\frac{1}{2}\text{C}-f\sin^2\frac{1}{2}\text{C}}=\frac{c+2\cot\frac{1}{2}\text{C}}{\cot^2\frac{1}{2}\text{C}-f}=\gamma.$$

The most general case solved by Mr. Lowry, in the Repository, is that in which $e=d$; that is, where any two of the three co-efficients d, e, f are equal. Then the preceding equation, when the values of β, α are written in it for this case, becomes

$$\frac{b+2\cot\frac{1}{2}\text{B}}{\cot^2\frac{1}{2}\text{B}-d}\cdot\frac{\cot^2\frac{1}{2}\text{A}-d}{a+2\cot\frac{1}{2}\text{A}}=\gamma;$$

and presents a very simple solution, for we may take $\text{B}=\pm\text{A}$, and it gives

$$\frac{b\pm 2\cot\frac{1}{2}\text{A}}{a+2\cot\frac{1}{2}\text{A}}=\gamma,\qquad\text{or }\cot\tfrac{1}{2}\text{A}=\frac{b-a\gamma}{2(\gamma\mp 1)};$$

so that

$$\frac{y}{x}=\alpha=\frac{4(b\mp a)(\gamma\mp 1)}{(b-a\gamma)^2-4d(\gamma\mp 1)^2},$$

$$\frac{z}{y}=\gamma=\frac{c+2\cot\frac{1}{2}\text{C}}{\cot^2\frac{1}{2}\text{C}-f}.$$

Hence we may take

$$x=t\{(b-a\gamma)^2-4d(\gamma\mp 1)^2\},$$

$$y=4t(b\mp a)(\gamma\mp 1),$$

$$z=4t(b\mp a)(\gamma\mp 1)\gamma.$$

Example 1. Let $a=b=c=-1$, $d=e=f=1$. The lower signs must be used, and $\cot\frac{1}{2}\text{C}$ must be taken either >1, or between $\frac{1}{2}$ and -1. If $\cot\frac{1}{2}\text{C}=4$, $t=-\frac{225}{16}$; then

$$x=117,\quad y=165,\quad z=77.$$

Example 2. Let $a = b = c = 1$, $d = e = f = -1$. The lower signs must be used, and we must have $\cot \frac{1}{2}c > -\frac{1}{2}$. If $\cot \frac{1}{2}c = 1$, and $t = 4$; then

$$x = 101, \qquad y = 80, \qquad z = 120.$$

PROBLEM XXVII. Find four numbers such that the sum of every two of them may be a square; the difference of every two of them, increased by a square which is to be found, may be a square; and the sum of all the four numbers, diminished by the aforesaid square, may be a square number.

Baker, Gents. Diary, Quest. 1360.

Solution. If s^2 is the square to be found, the equations are

$$v + x = a^2, \qquad v - x + s^2 = g^2,$$
$$v + y = b^2, \qquad v - y + s^2 = h^2,$$
$$v + z = c^2, \qquad v - z + s^2 = i^2,$$
$$x + y = d^2, \qquad x - y + s^2 = k^2,$$
$$x + z = e^2, \qquad x - z + s^2 = l^2,$$
$$y + z = f^2, \qquad y - z + s^2 = m^2,$$
$$v + x + y + z - s^2 = n^2.$$

The values of v, x, y, z, will be, as in Prob. VII.,

$$2v = a^2 + b^2 - d^2, \qquad 2x = a^2 - b^2 + d^2$$
$$2y = - a^2 + b^2 + d^2, \qquad 2z = c^2 + f^2 - b^2;$$

and the equations, after v, x, y, z are eliminated, are

$$a^2 + f^2 = b^2 + e^2 = c^2 + d^2 = n^2 + s^2,$$
$$a^2 + g^2 = b^2 + h^2 = c^2 + i^2,$$
$$a^2 + s^2 = b^2 + k^2 = c^2 + l^2,$$
$$b^2 + c^2 = n^2 + g^2, \qquad b^2 + s^2 = m^2 + c^2.$$

A general solution of these equations appears to be unattainable. The particular case discussed in the Diary, is that in which the square s^2 is assumed equal to one of the six squares, a^2, b^2, c^2 . . . . Thus, if

$$s = f, \text{ then } h = c,\ i = b,\ k = e,\ l = d,\ n = a\,;$$

and the eliminated equations are

$$a^2 + f^2 = b^2 + e^2 = c^2 + d^2,$$

$$a^2 + g^2 = b^2 + c^2,$$

$$a^2 + m^2 = b^2 + d^2.$$

Take, to solve the first three equations, by Prob. II.,

$$b = c \sin A + d \cos A, \qquad e = c \cos A - d \sin A,$$

$$a = c \sin B + d \cos B, \qquad f = c \cos B - d \sin B,$$

$$a = c \sin C + b \cos C, \qquad g = c \cos C - b \sin C.$$

By eliminating a and b between the first column of these equations, we find

$$\frac{c}{d} = \frac{\cos B - \cos A \cos C}{\sin C - \sin B + \sin A \cos C};$$

so that we may take

$$c = t\,(\cos B - \cos A \cos C),$$

$$d = t\,(\sin C - \sin B + \sin A \cos C),$$

$$a = t\{\cos B \sin C + \cos C \sin (A - B)\},$$

$$b = t\{\cos A \sin C + \sin (A - B)\}.$$

Substitute these in the fourth equation,

$$m^2 = b^2 + d^2 - a^2,$$

ordering the terms according to the functions of c; then

$$m^2=4t^2\{\mu^2\cos^2\tfrac{1}{2}\text{c}+\nu^2\sin^2\tfrac{1}{2}\text{c}+(p\cos^2\tfrac{1}{2}\text{c}+q\sin^2\tfrac{1}{2}\text{c})\sin\text{c}+r\sin^2\tfrac{1}{2}\text{c}\},$$

where we have put

$$\alpha = \tfrac{1}{2}(\text{A}+\text{B}), \qquad \beta = \tfrac{1}{2}(\text{A}-\text{B}),$$

$$\mu = \cos\alpha\sin\beta, \qquad \nu = \sin\alpha\cos\beta,$$

$$p = \sin\beta(\cos\alpha-\sin\alpha\sin 2\beta), \quad q = -\cos\beta(\sin\alpha-\cos\alpha\sin 2\beta),$$

$$4r = \cos^2(\alpha+\beta)+\sin^2 2\beta-\sin 2\alpha\sin 2\beta.$$

Then by Prob. XIII., Case 1,

$$m = 2t\left(\mu\cos^2\tfrac{1}{2}\text{c}\pm\nu\sin^2\tfrac{1}{2}\text{c}+\frac{p}{2\mu}\sin\text{c}\right),$$

$$\text{I.}\quad \tan\tfrac{1}{2}\text{A} = -\frac{\sin\beta\cos 2\alpha}{\cos\alpha},$$

$$\text{II.}\quad \tan\tfrac{1}{2}\text{A} = -\frac{\sin\beta}{\cos\alpha}-\frac{\sin^3\alpha}{\cos\beta\cos 2\alpha};$$

or,

$$m = 2t\left(\mu\cos^2\tfrac{1}{2}\text{c}\pm\nu\sin^2\tfrac{1}{2}\text{c}\pm\frac{q}{2\nu}\sin\text{c}\right),$$

$$\text{III.}\quad \cot\tfrac{1}{2}\text{A} = -\frac{\cos\beta\cos 2\alpha}{\sin\alpha},$$

$$\text{IV.}\quad \cot\tfrac{1}{2}\text{A} = \frac{\cos\beta}{\sin\alpha}-\frac{\cos^3\alpha}{\sin\beta\cos 2\alpha}.$$

By using the solution (I), we shall find, by putting

$$t = t'\left(\frac{\cos^2\alpha}{\sin\beta}+\sin\beta\cos^2 2\alpha\right),$$

$$a = t'\{\cos\beta\sin^2 2\alpha-\sin\beta\cos 2\alpha(\sin 2\alpha+\sin 2\beta\cos 2\alpha)\},$$

$$b = t'\{\cos\beta\sin^2 2\alpha+\sin\beta\cos 2\alpha(\sin 2\alpha+\sin 2\beta\cos 2\alpha)\},$$

$$c = t'\cos\alpha(\sin 2\alpha+\sin 2\beta\cos^2 2\alpha),$$

$$d = t'\sin\alpha(\sin 2\alpha-\sin 2\beta\cos^2 2\alpha).$$

These values, although very simple, are yet too complicated to enable us to determine, generally, the limits within which α and β must be taken, so as to produce positive numbers; but if we take $\alpha = \varphi(2)$, we shall easily find that sin 2β must be within the limits of 1 and ,9 nearly. Thus, for any given value of α, the limits for β may be ascertained.

Example. If $\alpha = \varphi(2), \beta = \varphi(\frac{7}{3}), t' = \dfrac{5^4 . 29^3}{96}$; we get

$a = 122823, \quad b = 79017, \quad c = 98919, \quad d = 111998;$

$v = 4392811807, \quad x = 10692677522,$

$y = 1850874482, \quad z = 5392156754.$

The general value of m^2, in this solution, is not a symmetrical function of A, B, C; so that, by ordering this square according to the functions of A, or of B, instead of those of C, we should obtain eight new and independent formulas for the question. Among these we may notice

$$\text{V. } \tan \tfrac{1}{2}B = \sin C + \frac{\cos C - \cos A}{\sin A + \cos A \sin C},$$

$$\text{VI. } \cot \tfrac{1}{2}B = \sin C + \frac{\cos C + \cos A}{\sin A - \cos A \sin C},$$

$$\text{VII. } \tan \tfrac{1}{2}(90^\circ - A) = \frac{(\sin C + \cos C - \sin B)^2 - \cos^2 B \sin C \cos C}{\cos B \cos C - \sin B \cos B \cos C(\sin C + \cos C)},$$

$$\text{VIII. } \cot \tfrac{1}{2}(90^\circ - A) = \frac{(\sin C - \cos C - \sin B)^2 + \cos^2 B \sin C \cos C}{\cos B \cos C - \sin B \cos B \cos C(\sin C - \cos C)}.$$

PROBLEM XXVIII.. To find five numbers such that the sum of every three of them is a square number.

Solution. The equations representing the question are

$$v+w+x=a^2, \qquad v+y+z=f^2,$$

$$v+w+y=b^2, \qquad w+x+y=g^2,$$

$$v+w+z=c^2, \qquad w+x+z=h^2,$$

$$v+x+y=d^2, \qquad w+y+z=i^2,$$

$$v+x+z=e^2, \qquad x+y+z=k^2.$$

The values of v, x, &c., in these equations are

$$3v=a^2+b^2+d^2-2g^2, \quad 3x=a^2-2b^2+d^2+g^2,$$

$$3w=a^2+b^2-2d^2+g^2, \quad 3y=-2a^2+b^2+d^2+g^2,$$

$$3z=-2a^2+c^2+e^2+h^2;$$

and the equations, after v, w, &c. are eliminated, are

$$a^2+f^2=b^2+e^2=c^2+d^2,$$

$$a^2+i^2=b^2+h^2=c^2+g^2,$$

$$k^2=g^2+f^2-b^2.$$

The four first of these are solved by taking

$$b=a\sin\text{A}-f\cos\text{A}, \qquad e=a\cos\text{A}+f\sin\text{A},$$

$$c=a\sin\text{B}-f\cos\text{B}, \qquad d=a\cos\text{B}+f\sin\text{B},$$

$$b=a\sin\text{C}-i\cos\text{C}, \qquad h=a\cos\text{C}+i\sin\text{C},$$

$$c=a\sin\text{D}-i\cos\text{D}, \qquad g=a\cos\text{D}+i\sin\text{D}.$$

The equal values of b and c give

$$a(\sin A - \sin C) = f \cos A - i \cos C,$$

$$a(\sin B - \sin D) = f \cos B - i \cos D;$$

and, eliminating a,

$$\frac{f}{i} = \frac{\sin B \cos C - \sin A \cos D + \sin(C - D)}{\cos B \sin C - \cos A \sin D - \sin(A - B)};$$

so that we may take

$$i = t\{\cos A \sin D - \cos B \sin C + \sin(A - B)\},$$

$$f = t\{\sin A \cos D - \sin B \cos C - \sin(C - D)\},$$

$$a = t\{\cos A \cos D - \cos B \cos C\},$$

$$b = t\{\cos A \sin(C - D) - \cos C \sin(A - B)\},$$

$$c = t\{\cos B \sin(C - D) - \cos D \sin(A - B)\},$$

$$d = t\{-\cos C - \sin B \sin(C - D) + \cos D \cos(A - B)\},$$

$$e = t\{\quad \cos D - \sin A \sin(C - D) - \cos C \cos(A - B)\},$$

$$g = t\{\quad \cos A - \cos B \cos(C - D) + \sin D \sin(A - B)\},$$

$$h = t\{-\cos B + \cos A \cos(C - D) + \sin C \sin(A - B)\}.$$

Substituting the values of g, f, b in the fifth equation, and arranging the results according to the functions of A, it becomes

$$k^2 = t^2\{\mu^2 \cos^2\tfrac{1}{2}A + \nu^2 \sin^2\tfrac{1}{2}A + (p \cos^2\tfrac{1}{2}A + q \sin^2\tfrac{1}{2}A)\sin A + r \sin^2 A\},$$

in which,

$$\mu = 1 - \sin B \sin D - \cos B \cos (C - D),$$

$$\nu = 1 - \sin B \sin D + \cos B \cos (C - D),$$

$$p = 2\mu \cos B \sin D + 2s (\cos B \cos C - \cos D),$$

$$q = -2\nu \cos B \sin D - 2s (\cos B \cos C + \cos D),$$

$$r = 2 \sin B \sin D - 2 \sin^2 B \sin^2 D - \cos^2 B \cos^2 C + s^2,$$

$$s = \sin B \cos C + \sin (C - D).$$

Then, by Prob. XIII., Case 1, we may have

$$k = t (\mu \cos^2 \tfrac{1}{2}A + \nu \sin^2 \tfrac{1}{2}A + \frac{p}{2\mu} \sin A),$$

$$\text{I. } \tan \tfrac{1}{2}A = \frac{4r\mu^2 + (\mu - \nu)^2 \mu^2 - p^2}{2\mu(p\nu - q\mu)} = \frac{(\cos B \cos C - \cos D)^2 - \mu^2}{2\mu \cos B \sin D};$$

or $$k = t (\mu \cos^2 \tfrac{1}{2}A + \nu \sin^2 \tfrac{1}{2}B + \frac{q}{2\nu} \sin A),$$

$$\text{II. } \cot \tfrac{1}{2}A = \frac{4r\nu^2 + (\mu - \nu)^2 \nu^2 - q^2}{2\nu(q\mu - p\nu)} = \frac{\nu^2 - (\cos B \cos C + \cos D)^2}{2\nu \cos B \sin D}.$$

To complete this solution it would be necessary to determine the limits within which the angles B, C, D may be assumed, so that the resulting values of v, w, x, &c., may be positive numbers; but the complexity of the formulas do not encourage the attempt, and the object seems unworthy of the effort, which is one of mere labor. The following examples are taken at hazard.

Example 1. If $C = \varphi(3)$, $D = 180° - C$, $B = \varphi(\tfrac{7}{3})$, then (II.) gives $\cot \tfrac{1}{2}A = \tfrac{1}{4}$; and taking $t = \tfrac{1}{2} . 25 . 17 . 29$, we have for the roots of the squares, which may all be taken positive,

$$a = 950, \quad b = 470, \quad c = 670, \quad d = 674, \quad e = 826,$$

$$f = 26, \quad g = 685, \quad h = 835, \quad i = 125, \quad k = 499;$$

and for the numbers

$$v = 213075\tfrac{1}{3}, \quad w = 228024\tfrac{1}{3}, \quad x = 461400\tfrac{1}{3},$$

$$y = -220199\tfrac{2}{3}, \quad z = 7800\tfrac{1}{3};$$

which may be multiplied by 9 to produce integers.

Example 2. If $C = \varphi(3)$, $D = 180° - C$, $B = \varphi(\frac{3}{2})$, then (II.) gives $\cot \frac{1}{2}A = -\frac{9}{11}$; and taking $t = \dfrac{25 \,.\, 13 \,.\, 101}{14}$, we find

$$a = 350, \quad b = 810, \quad c = 1230, \quad d = 2042, \quad e = 2242,$$

$$f = 2358, \quad g = 485, \quad h = 1045, \quad i = 1275, \quad k = 2267;$$

and the numbers are

$$v = 1492638, \quad w = -2441901, \quad x = 1071763,$$

$$y = 1605363, \quad z = 2462163.$$

From these two examples, it is probable that, C and D remaining the same, positive sets of numbers would result for some values of B between $\varphi(\frac{3}{2})$ and $\varphi(\frac{7}{3})$.

Cor. The squares $a^2, b^2, c^2 \ldots . k^2$, having the relations established in the preceding solution, if we take

$$2v = -a^2 + b^2 + c^2, \; 2w = a^2 - b^2 + c^2, \; 2x = a^2 + b^2 - c^2,$$

$$2y = -a^2 + d^2 + e^2, \quad 2z = -a^2 + g^2 + h^2;$$

we should have, by substituting these relations,

$$v + w = c^2, \quad v + z = i^2, \quad w + z = h^2,$$

$$v + x = b^2, \quad w + x = a^2, \quad x + y = d^2,$$

$$v + y = f^2, \quad w + y = e^2, \quad x + z = g^2,$$

$$y + z = k^2,$$

that is, the sum of every two of the numbers v, w, x, y, z would be a square number, and they would answer the question proposed by Mr. Baker, in the Gentleman's Diary for 1838: "To find five numbers, the sum of every two of which is a square number."

Example. By taking $a, b, c, \ldots. k$ as in the first example, we shall have

$$v = -116350, \qquad w = 565250, \qquad x = 337250,$$
$$y = 117026, \qquad z = 131975.$$

PROBLEM XXIX. To find n numbers such that, if the square of each of them be subtracted from the square of their sum, the several remainders may be rational squares.

Solution. If z, y, x, &c. be the n numbers, and s their sum, the equations which represent the question may evidently be put in the form

$$(z+y+x+\&c.)^2 = s^2 = a^2+z^2 = b^2+y^2 = c^2+x^2 = \&c.;$$

which correspond to the General Case of Prob. X., when $r = 2$; that is, we may take

$$z = s \cos A, \qquad y = s \cos B, \qquad x = s \cos C, \ \&c.,$$

and the final equation (b), of Prob. X. will be

$$\Sigma \cos A = 1.$$

Now take

$$B = A - \beta, \qquad C = A - \gamma, \qquad D = A - \delta, \ \&c.,$$

and this equation becomes

$$\cos A\,(1 + \Sigma \cos\beta) + \sin A\, \Sigma \sin\beta = 1. \quad . \quad (a).$$

First. Let n be an odd number, so that the number of

angles β, γ, δ, &c. will be even. Then if these angles be taken in supplementary pairs; that is, if

$$\beta + \gamma = \delta + \varepsilon = \&c. = 180^\circ;$$

we shall have

$$\Sigma \cos \beta = 0,$$

and equation (a) will give

$$\tan \tfrac{1}{2}A = \Sigma \sin \beta.$$

Case 1. Let $n = 3$, then the number of angles β, γ, &c. is two, and since $\gamma = 180^\circ - \beta$,

$$\tan \tfrac{1}{2}A = 2 \sin \beta.$$

Then we may have

$$s = t\,(1 + 4 \sin^2\beta),$$

$$z = s \cos A = t\,(1 - 4 \sin^2\beta),$$

$$y = s \cos(A - \beta) = t\,\{4 \sin^2\beta + \cos \beta(1 - 4 \sin^2\beta)\},$$

$$x = -s \cos(A + \beta) = t\,\{4 \sin^2\beta - \cos \beta(1 - 4 \sin^2\beta)\};$$

in which, to make the numbers positive, it is necessary to have

$$\cot \tfrac{1}{2}\beta > 3{,}732 < 5{,}568.$$

Example. If $\beta = \varphi(5)$ and $t = 13^3$, we have

$$z = 897, \qquad y = 2128, \qquad x = 472.$$

This case was proposed by Mr. William Wright in the Mathematical Companion, No. 28, Quest. 20.

Case 2. Let $n = 5$; the number of angles will be four, and we must have

$$\gamma = 180^\circ - \beta, \qquad \varepsilon = 180^\circ - \delta;$$

$$\tan \tfrac{1}{2}A = 2 \sin \beta + 2 \sin \delta = u;$$

so that we may take

$$s = t(1 + u^2),$$

$$z = s \cos A = t(1 - u^2),$$

$$y = s \cos(A - \beta) = t\{2u \sin\beta + \cos\beta(1 - u^2)\},$$

$$x = -s \cos(A + \beta) = t\{2u \sin\beta - \cos\beta(1 - u^2)\},$$

$$w = s \cos(A - \delta) = t\{2u \sin\delta + \cos\delta(1 - u^2)\},$$

$$v = -s \cos(A + \delta) = t\{2u \sin\delta - \cos\delta(1 - u^2)\};$$

where the angles β, δ must be so assumed that we may have

$$u < 1 > \tfrac{1}{2} \cot\delta(1 - u^2,)$$

δ being the less one of the two angles.

Example If $\beta = \varphi(7)$, $\delta = \varphi(9)$, $t = 1025^3$, we have

$z = 2100225$, $y = 604486616$, $x = 600454184$,

$w = 474369000$, $v = 470271000$.

Second. Let n be an even number, so that the number of angles β, γ, δ, &c. will be odd.

Let A', B', C', &c. be an odd number of angles, found as in the preceding part of the solution, such that

$$\Sigma \cos A' = 1,$$

Then take

$$\beta + A' = \gamma + B' = \delta + C' = \&c. = 90^\circ;$$

then will

$$\Sigma \sin\beta = \Sigma \cos A' = 1,$$

and equation (a) becomes

$$\cos A\,(1 + \Sigma \sin A') + \sin A = 1.$$

$$\tan \tfrac{1}{2}(90^\circ - A) = 1 + \Sigma \sin A',$$

$$\cot \tfrac{1}{2}A = -\frac{2 + \Sigma \sin A'}{\Sigma \sin A'}.$$

The analytical solution of the question is thus completely effected.

Case 3. Let $n = 4$, and the number of angles A', B', C' will be three; then by Case 1, we have for these angles

$$\cos A' = \frac{1 - 4 \sin^2\beta}{1 + 4 \sin^2\beta},$$

$$\sin A' = \frac{4 \sin \beta}{1 + 4 \sin^2\beta},$$

$$\cos B' = \frac{4 \sin^2\beta + \cos\beta\,(1 - 4 \sin^2\beta)}{1 + 4 \sin^2\beta},$$

$$\sin B' = \frac{2 \sin 2\beta - \sin\beta\,(1 - 4 \sin^2\beta)}{1 + 4 \sin^2\beta},$$

$$\cos C' = \frac{4 \sin^2\beta - \cos\beta(1 - 4 \sin^2\beta)}{1 + 4 \sin^2\beta},$$

$$\sin C' = \frac{-2 \sin 2\beta - \sin\beta(1 - 4 \sin^2\beta)}{1 + 4 \sin^2\beta};$$

$$\Sigma \cos A' = 1, \qquad \Sigma \sin A' = 2 \sin\beta,$$

$$\cot \tfrac{1}{2}A = -\frac{1 + \sin\beta}{\sin\beta}.$$

Then if we take

$$s = t(1 + 2\sin\beta + 2\sin^2\beta)(1 + 4\sin^2\beta);$$

$$z = s\cos A \quad = t(1 + 2\sin\beta)(1 + 4\sin^2\beta),$$

$$y = s\sin(A+A') = 2t\sin\beta(1 + 2\sin\beta)(1 + \sin\beta + 2\sin^2\beta),$$

$$x = s\sin(A+B') = t\sin\beta\{2\cos\beta(1+2\sin\beta)(1+\sin\beta+2\sin^2\beta)$$
$$-1-2\sin\beta-4\sin^2\beta\},$$

$$w = s\sin(A+C') = t\sin\beta\{-2\cos\beta(1+2\sin\beta)(1+\sin\beta+2\sin^2\beta)$$
$$-1-2\sin\beta-4\sin^2\beta\};$$

which numbers answer the conditions analytically, but one of them, at least, must be numerically negative.

Example. Let $\beta = \varphi(3)$, $t = 5^5$; we shall have

$$z = 16775, \quad y = 19140, \quad x = 8487, \quad w = -22137, \quad s = 22265,$$

which satisfy the equations

$$s^2 = a^2 + z^2 = b^2 + y^2 = c^2 + x^2 = d^2 + w^2.$$

Having found one set of angles, A, B, C, &c. making

$$\Sigma \cos A = 1,$$

another set, A′, B′, C′, &c. may easily be deduced from them, having the same property; for if we put $A' = A - \varphi$, $B' = B - \varphi$, $C' = C - \varphi$, &c.; then

$$\Sigma \cos A' = \cos\varphi\,\Sigma\cos A + \sin\varphi\,\Sigma\sin A$$
$$= \cos\varphi + \sin\varphi\,\Sigma\sin B = 1;$$
$$\tan\tfrac{1}{2}\varphi = \Sigma\sin A.$$

Probably positive numbers might be obtained from this process, but they would be too large to calculate to advantage.

Problem XXX. To find n square numbers such that the sum of every $n-1$ of them may be square numbers.

Solution. The question is obviously represented by the equations

$$z^2 + y^2 + x^2 + \&c. = a^2 + z^2 = b^2 + y^2 = c^2 + x^2 = \&c.,$$

which are those of Problem IX. Then we may take

$$y = a \sin A - z \cos A,$$

$$x = a \sin B - z \cos B,$$

$$\&c.$$

and equation (a) of that Problem becomes

$$y^2 + x^2 + \&c. = a^2,$$

or, by substitution,

$$z^2 \Sigma \cos^2 A - az\Sigma \sin 2A = a^2(1 - \Sigma \sin^2 A).$$

Now put

$$z\Sigma \cos^2 A - \tfrac{1}{2}a\Sigma \sin 2a = \pm ka, \qquad (a)$$

then we shall have, there being $n-1$ of the angles A, B, &c.,

$$4k^2 = 4\Sigma \cos^2 A - 4\Sigma \cos^2 A \,.\, \Sigma \sin^2 A + (\Sigma \sin 2A)^2$$

$$= 2\Sigma(1+\cos 2A) - \Sigma(1+\cos 2A)\,.\,\Sigma(1-\cos 2A) + (\Sigma \sin 2A)^2$$

$$= 2(n-1) - (n-1)^2 + 2\Sigma \cos 2A + (\Sigma \cos 2A)^2 + (\Sigma \sin 2A)^2$$

$$= 3(n-1) - (n-1)^2 + 2\Sigma \cos 2A + 2\Sigma \cos 2(A-B)\,; \qquad (b)$$

$$2k^2 = \tfrac{1}{2}(n-1)(4-n) + \Sigma \cos 2A + \Sigma \cos 2(A-B),$$

$$k^2 = k_1{}^2 + \sin A \cos A \Sigma \sin 2B - (1 + \Sigma \cos 2B) \sin^2 A\,;$$

in which

$$2k_1^2 = \tfrac{1}{2}(n-1)(4-n) + 1 + 2\,\Sigma \cos 2\text{B} + \Sigma \cos 2(\text{B}-\text{C}),$$

$$k_1^2 = k_2^2 + \sin \text{B} \cos \text{B}\, \Sigma \sin 2\text{C} - (2 + \Sigma \cos 2\text{C}) \sin^2 \text{B};$$

in which

$$2k_2^2 = \tfrac{1}{2}(n-1)(4-n) + 3 + 3\,\Sigma \cos 2\text{C} + \Sigma \cos 2(\text{C}-\text{D}),$$

$$k_2^2 = k_3^2 + \sin \text{C} \cos \text{C}\, \Sigma \sin 2\text{D} - (3 + \Sigma \cos 2\text{D}) \sin^2 \text{C};$$

in which

$$2k_3^2 = \tfrac{1}{2}(n-1)(4-n) + 6 + 4\,\Sigma \cos 2\text{D} + \Sigma \cos 2(\text{D}-\text{E}),$$

$$k_3^2 = k_4^2 + \sin \text{D} \cos \text{D}\, \Sigma \sin 2\text{E} - (4 + \Sigma \cos 2\text{E}) \sin^2 \text{D};$$

&c. &c.

It is understood that $\Sigma \sin 2\text{B}$ represents the sum of the sines of all the double angles, except A; $\Sigma \sin 2\text{C}$, the sum of the sines of all the double angles, except A and B; &c. &c. In general, it is evident, from the law of continuity of these equations, that

$$2k_i^2 = \tfrac{1}{2}(n-1)(4-n) + \tfrac{1}{2}i(i+1) + (i+1)\Sigma \cos 2\text{L} + \Sigma \cos 2(\text{L}-\text{M}),$$

and there would be left $n-i-1$ of the angles L, M, &c.; so that, when

$$i = n-3,$$

there would only be two of them, say Y and Z, left. Thus, for this equation, we have

$$\begin{aligned} 2k_{n-3}^2 &= \tfrac{1}{2}(n-1)(4-n) + \tfrac{1}{2}(n-3)(n-2) + (n-2)(\cos 2\text{Y} + \cos 2\text{Z}) \\ &\qquad + \cos 2(\text{Y}-\text{Z}) \\ &= 1 + (n-2)(\cos 2\text{Y} + \cos 2\text{Z}) + \cos 2(\text{Y}-\text{Z}), \end{aligned}$$

$$k_{n-3}^2 = \cos^2(\text{Y}-\text{Z}) + (n-2)\cos(\text{Y}+\text{Z})\cos(\text{Y}-\text{Z}).$$

A simple solution of this equation is

$$Y + Z = 90°,$$

$$k_{n-3} = \cos(Y - Z) = \sin 2Y.$$

The preceding equations all belong to the form of Problem XIII., Case 5, and therefore we may put

$$k = k_1 - \frac{\Sigma \sin 2B}{2k_1} . \sin A,$$

$$k_1 = k_2 - \frac{\Sigma \sin 2C}{2k_2} . \sin B,$$

&c. &c.

$$k_{n-4} = k_{n-3} - \frac{\Sigma \sin 2Y}{2k_{n-3}} . \sin X$$

$$= \sin 2Y - \sin X;$$

then we shall have

$$\cot \tfrac{1}{2}A = \frac{1 + \Sigma \cos 2B}{\Sigma \sin 2B} + \frac{\Sigma \sin 2B}{4k_1{}^2},$$

$$\cot \tfrac{1}{2}B = \frac{2 + \Sigma \cos 2C}{\Sigma \sin 2C} + \frac{\Sigma \sin 2C}{4k_2{}^2},$$

$$\cot \tfrac{1}{2}C = \frac{3 + \Sigma \cos 2D}{\Sigma \sin 2D} + \frac{\Sigma \sin 2D}{4k_3{}^2},$$

&c.

$$\cot \tfrac{1}{2}X = \frac{n-2}{2 \sin 2Y};$$

which completely resolves the Problem.

Case 1. Let $n = 3$, and there are two angles A and B; the equations become

$$A + B = 90^\circ,$$

$$k = \sin 2A;$$

$$\Sigma \cos^2 A = 1,$$

and equation (a) becomes

$$z = 2a \sin 2A,$$

so that the solution is precisely that given for this particular case in Problem XXII.

Case 2. Let $n = 4$; there are three angles, A, B, C; and the equations are

$$B + C = 90^\circ,$$

$$\cot \tfrac{1}{2}A = \frac{1}{\sin 2B},$$

$$k = \sin 2B - \sin A = \frac{\sin 2B \cos^2 2B}{1 + \sin^2 2B};$$

and equation (a) becomes

$$\frac{z}{a} = \sin 2B, \quad \text{or} = \frac{2 \sin 2B}{1 + \sin^4 2B}.$$

But the first of these values makes $y = z$; and taking the second, and making

$$a = t(1 + \sin^4 2B),$$

we get for the numbers

$$z = 2t \sin 2B, \qquad x = t \sin B \cos 2B (4 \cos^2 B - \cos^3 2B),$$

$$y = 2t \sin^3 2B, \qquad w = t \cos B \cos 2B (4 \sin^2 B + \cos^3 2B).$$

Example. If $B = \varphi(2)$, $t = 5^5$; the numbers are

3750000, 3456000, 639604, 832797.

But, in this particular case, equation (b) takes a form susceptible of particular reductions; thus

$$2k^2 = \Sigma \cos 2A + \Sigma \cos 2(A - B),$$

$$k^2 = \cos(A+B-C)\cos(A-B+C) + \cos(A+B-C)\cos(-A+B+C)$$
$$+ \cos(A-B+C)\cos(-A+B+C).$$

Then if we take

$$A + B - C = 90^\circ, \qquad \text{or} \quad C = A + B - 90^\circ;$$

$$k^2 = \sin 2A \sin 2B, \quad . \quad . \quad . \quad (c)$$

which, like the final equation of Problem XXIII., to which it is analogous, admits of many solutions.

1st. By dividing it by $4 \cos^4 \frac{1}{2}B$, it becomes

$$\tfrac{1}{4}k^2 \sec^4 \tfrac{1}{2}B = \sin 2A \tan \tfrac{1}{2}B \,(1 - \tan^2 \tfrac{1}{2}B);$$

so that if we take

$$\tan \tfrac{1}{2}B = \sin 2A;$$

$$\tfrac{1}{2}k \sec^2 \tfrac{1}{2}B = \sin 2A \cos 2A,$$

$$k = \frac{\sin 4A}{1 + \sin^2 2A};$$

and equation (a) becomes

$$\frac{z}{a} = \frac{\sin 2A(1 - 2\cos 2A + \sin^2 2A)}{1 + \sin^2 2A + 2\cos 2A \sin^2 2A}.$$

Then we may take

$$a = t\,(1 + \sin^2 2A + 2\cos 2A \sin^2 2A),$$

$$z = t \sin 2A\,(1 - 2\cos 2A + \sin^2 2A),$$

$$y = t \sin A \cos 2A\,(1 + 2\cos 2A + \sin^2 2A),$$

$$x = t \sin 2A\,(1 + 2\cos 2A + \sin^2 2A),$$

$$w = t \cos A \cos 2A\,(1 - 2\cos 2A + \sin^2 2A).$$

Example. If $A = \varphi(2)$, $t = 5^7$; the numbers are

186120, 23828, 102120, 32571.

2d. By dividing equation (c) by $16 \cos^4 \frac{1}{2}A \cos^4 \frac{1}{2}B$, it gives

$$\tfrac{1}{16}k^2 \sec^4 \tfrac{1}{2}A \sec^4 \tfrac{1}{2}B = \tan\tfrac{1}{2}A \tan\tfrac{1}{2}B(1 - \tan^2\tfrac{1}{2}A)(1 - \tan^2\tfrac{1}{2}B).$$

In this equation, put

$$\tan\tfrac{1}{2}B = 1 - \tan\tfrac{1}{2}A;$$

then it becomes

$$\tfrac{1}{16}k^2\sec^4\tfrac{1}{2}A\sec^4\tfrac{1}{2}B = \tan^2\tfrac{1}{2}A(1-\tan\tfrac{1}{2}A)^2\times(1+\tan\tfrac{1}{2}A)(2-\tan\tfrac{1}{2}A).$$

Now, let

$$1 + \tan\tfrac{1}{2}A = m\cot\tfrac{1}{2}\theta, \qquad 2 - \tan\tfrac{1}{2}A = m\tan\tfrac{1}{2}\theta,$$

so that

$$\tfrac{1}{16}k^2 \sec^4\tfrac{1}{2}A \sec^4\tfrac{1}{2}B = \tan^2\tfrac{1}{2}A\,(1 - \tan\tfrac{1}{2}A)^2 \times m^2;$$

then we shall find

$$3 = m(\cot\tfrac{1}{2}\theta + \tan\tfrac{1}{2}\theta), \qquad \text{or} \quad m = \tfrac{3}{2}\sin\theta;$$

$$\tan\tfrac{1}{2}A = \tfrac{1}{2}(1 + 3\cos\theta), \qquad \tan\tfrac{1}{2}B = \tfrac{1}{2}(1 - 3\cos\theta);$$

$$k = \frac{24\sin\theta\,(1 - 9\cos^2\theta)}{(5 + 9\cos^2\theta)^2 - 36\cos^2\theta}.$$

By substituting in equation (a), we obtain, after some reduction

$$\frac{a}{z}=\frac{3}{4}\cdot\frac{32+24\sin^2\theta\cos^2\theta-3\sin^4\theta\mp4\sin\theta(7+9\cos^2\theta)}{(1-9\cos^2\theta)(4\mp3\sin\theta)};$$

so that we may take

$$a=3t\{32+24\sin^2\theta\cos^2\theta-3\sin^4\theta\mp4\sin\theta(7+9\cos^2\theta)\},$$

$$z=4t(1-9\cos^2\theta)(4\mp3\sin\theta),$$

$$y=12t(1+3\cos\theta)\{4\sin^2\tfrac{1}{2}\theta+3\sin^2\theta\mp\sin\theta(5-3\cos\theta)\},$$

$$x=12t(1-3\cos\theta)\{4\cos^2\tfrac{1}{2}\theta+3\sin^2\theta\mp\sin\theta(5+3\cos\theta)\},$$

$$w=3t\sin\theta(1-9\cos^2\theta)(3\sin\theta\mp4).$$

Example. Let $\theta=\varphi(2)$, and $t=\frac{5^4}{32}$, then, when the upper signs are used, we obtain the numbers

$$z=280,\quad y=105,\quad x=60,\quad w=168;$$

and when the lower signs, the numbers

$$z=1120,\quad y=3465,\quad x=1980,\quad w=672.$$

3d. The equation

$$\tfrac{1}{16}k^2\sec^4\tfrac{1}{2}A\sec^4\tfrac{1}{2}B=\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B(1-\tan^2\tfrac{1}{2}A)(1-\tan^2\tfrac{1}{2}B),$$

may be solved by putting

$$\tan\tfrac{1}{2}A\,(1-\tan^2\tfrac{1}{2}A)=\tan\tfrac{1}{2}B\,(1-\tan^2\tfrac{1}{2}B),$$

or, $$\tan^2\tfrac{1}{2}A+\tan\tfrac{1}{2}A\tan\tfrac{1}{2}B+\tan^2\tfrac{1}{2}B=1.$$

Then, by Prob. IV., Ex. 1, we have

$$\tan\tfrac{1}{2}A=\frac{2\cos\theta}{2+\sin\theta},\qquad \tan\tfrac{1}{2}B=\frac{2\sin\theta+2\sin^2\tfrac{1}{2}\theta}{2+\sin\theta};$$

from which a solution similar to the preceding one may be obtained.

I believe that this case was first proposed and solved by Dr. O'Riordon, in No. 12 of the *Mathematical Companion.*

Case 3. Let $n=5$; there are four angles, A, B, C, D, and the equations for determining them are

$$\text{D}=90^\circ-\text{C},$$

$$\cot\tfrac{1}{2}\text{B}=\frac{3}{2\sin 2\text{C}},\qquad k_1=\sin 2\text{C}-\sin\text{B},$$

$$\cot\tfrac{1}{2}\text{A}=\frac{1+\Sigma\cos 2\text{B}}{\Sigma\sin 2\text{B}}+\frac{\Sigma\sin 2\text{B}}{4k_1{}^2}=\frac{2\sin^2\text{B}}{\sin 2\text{B}+2\sin 2\text{C}}+\frac{\sin 2\text{B}+2\sin 2\text{C}}{4k_1{}^2},$$

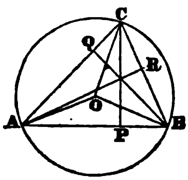
$$k=k_1-\frac{\Sigma\sin 2\text{B}}{k_1}.\sin\text{A}=k_1-\frac{\sin 2\text{B}+2\sin 2\text{C}}{2k_1}.\sin\text{A}.$$

The smallest numbers resulting from these formulas are, however, so very large as to discourage any attempt to calculate them. The roots of the squares, for instance, found by taking $\text{C}=\varphi(2)$, will probably not have fewer than thirty figures, and they *may* have more than sixty figures. Since an analytical solution has thus been obtained, modified forms, as in the preceding case, may possibly be met with, which shall contain less numerical results.

CHAPTER III.

APPLICATION TO GEOMETRY.

Theorem I. If the radius of the circle which circumscribes a plane triangle, and the trigonometrical functions of the angles of the triangle are expressed in rational numbers; then the sides of the triangle, its area, the perpendiculars on the sides from the opposite angles, and the segments into which these perpendiculars divide the sides, will all be expressed in rational numbers.



Demonstration. Let ABC be the triangle, O the centre of the circumscribing circle; R its radius; a, b, c the three sides, opposite the angles A, B, C; and S its area.

Then the angle AOB, at the centre $= 2\text{C}$, and the isosceles triangle AOB gives

$$c = 2\text{R} \sin \text{C};$$

similarly

$$b = 2\text{R} \sin \text{B},$$

$$a = 2\text{R} \sin \text{A};$$

which may all be included in the form

$$\frac{a}{\sin \text{A}} = \frac{b}{\sin \text{B}} = \frac{c}{\sin \text{C}} = 2\text{R}.$$

Again

$$S = \tfrac{1}{2}ab \sin C = 2R^2 \sin A \sin B \sin C;$$

$$CP = \quad b \sin A = 2R \quad \sin A \sin B;$$

that is

$$CP \sin C = BQ \sin B = AR \sin A = 2R \sin A \sin B \sin C.$$

Also $\quad AP = 2R \sin B \cos A, \qquad BP = 2R \sin A \cos B,$

and so for the other segments. Thus the truth of the theorem is established.

Example. By Chap. I., Problem III., Cor. 1; we may take

$$A = \varphi(2), \quad B = \varphi(\tfrac{3}{2}), \quad C = \varphi(\tfrac{7}{4});$$

and if $2R = \frac{65}{4}$, the sides are

$$a = 13, \qquad b = 15, \qquad c = 14;$$

the perpendiculars

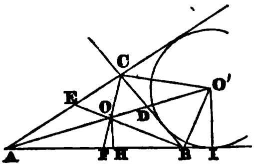
$$CP = 12, \qquad BQ = 11\tfrac{1}{5}, \qquad AR = 12\tfrac{12}{13};$$

and the segments

$$AP = 9, \qquad AQ = 8\tfrac{2}{5}, \qquad BR = 5\tfrac{5}{13};$$

and, if these numbers be multiplied by 65, they will be integers.

Theorem II. If the radius of the circle which circumscribes a plane triangle, and the trigonometrical functions of the *half angles* of the triangles are expressed in rational numbers; then the radii of the four circles touching the sides, the lines drawn from the angles to the centres of these circles, the lines bisecting the angles and terminating in the opposite sides, and the segments of the sides into which these lines divide the sides, will all be expressed in rational numbers.



Demonstration. Let

O be the centre of the inscribed circle,

$r = \text{OH}$, its radius,

O′ the centre of the circle touching a externally.

$r_a = \text{O}'\text{I}$, its radius,

OH, O′I perpendiculars to AB.

Since AO, BO bisect the angles A, B, we have

$$\text{AH} = r \cot \tfrac{1}{2}\text{A}, \qquad \text{BH} = r \cot \tfrac{1}{2}\text{B};$$

by addition

$$c = 2\text{R} \sin \text{C} = r\,(\cot \tfrac{1}{2}\text{A} + \cot \tfrac{1}{2}\text{B})$$

$$= r \,.\, \frac{\sin \frac{1}{2}(\text{A} + \text{B})}{\sin \frac{1}{2}\text{A} \sin \frac{1}{2}\text{B}}$$

$$= r \,.\, \frac{\cos \frac{1}{2}\text{C}}{\sin \frac{1}{2}\text{A} \sin \frac{1}{2}\text{B}},$$

$$r = 4\text{R} \sin \tfrac{1}{2}\text{A} \sin \tfrac{1}{2}\text{B} \sin \tfrac{1}{2}\text{C}.$$

Since BO′ bisects the angle CBI, we have

$$\text{AI} = r_a \cot \tfrac{1}{2}\text{A}, \qquad \text{BI} = r_a \tan \tfrac{1}{2}\text{B};$$

by subtraction

$$c = 2\text{R} \sin \text{C} = r_a\,(\cot \tfrac{1}{2}\text{A} - \tan \tfrac{1}{2}\text{B})$$

$$= r_a \,.\, \frac{\cos \frac{1}{2}(\text{A} + \text{B})}{\sin \frac{1}{2}\text{A} \cos \frac{1}{2}\text{B}}$$

$$= r_a \,.\, \frac{\sin \frac{1}{2}\text{C}}{\sin \frac{1}{2}\text{A} \cos \frac{1}{2}\text{B}},$$

$$r_a = 4\text{R} \sin \tfrac{1}{2}\text{A} \cos \tfrac{1}{2}\text{B} \cos \tfrac{1}{2}\text{C}.$$

Similarly, if r_b be the radius of the circle touching the three sides, b externally; and r_c the radius of the circle touching the three sides, c externally; we should have

$$r_a \cot \tfrac{1}{2}\text{A} = r_b \cot \tfrac{1}{2}\text{B} = r_c \cot \tfrac{1}{2}\text{C} = 4\text{R} \cos \tfrac{1}{2}\text{A} \cos \tfrac{1}{2}\text{B} \cos \tfrac{1}{2}\text{C}.$$

For the lines AO, BO, &c., we have

$$\begin{aligned}
\text{AO} &= r \ \text{cosec}\, \tfrac{1}{2}\text{A} = 4\text{R} \sin \tfrac{1}{2}\text{B} \sin \tfrac{1}{2}\text{C},\\
\text{BO} &= r \ \text{cosec}\, \tfrac{1}{2}\text{B} = 4\text{R} \sin \tfrac{1}{2}\text{A} \sin \tfrac{1}{2}\text{C},\\
\text{CO} &= r \ \text{cosec}\, \tfrac{1}{2}\text{C} = 4\text{R} \sin \tfrac{1}{2}\text{A} \sin \tfrac{1}{2}\text{B};\\
\text{AO}' &= r_a \,\text{cosec}\, \tfrac{1}{2}\text{A} = 4\text{R} \cos \tfrac{1}{2}\text{B} \cos \tfrac{1}{2}\text{C},\\
\text{BO}' &= r_a \ \sec \tfrac{1}{2}\text{B} = 4\text{R} \sin \tfrac{1}{2}\text{A} \cos \tfrac{1}{2}\text{C},\\
\text{CO}' &= r_a \ \sec \tfrac{1}{2}\text{C} = 4\text{R} \sin \tfrac{1}{2}\text{A} \cos \tfrac{1}{2}\text{B},\\
&\quad\&\text{c.} \qquad\qquad \&\text{c.}
\end{aligned}$$

For the lines bisecting the angles,

$$\frac{\text{AD}}{c} = \frac{\sin \text{B}}{\sin(\tfrac{1}{2}\text{A} + \text{B})} = \frac{\sin \text{B}}{\cos \tfrac{1}{2}(\text{B} - \text{C})},$$

or $$\text{AD} \cos \tfrac{1}{2}(\text{B} - \text{C}) = 2\text{R} \sin \text{B} \sin \text{C};$$

so $$\text{BE} \cos \tfrac{1}{2}(\text{A} - \text{C}) = 2\text{R} \sin \text{A} \sin \text{C},$$

$$\text{CF} \cos \tfrac{1}{2}(\text{A} - \text{B}) = 2\text{R} \sin \text{A} \sin \text{B}.$$

For the segments of the sides,

$$\frac{\text{AF}}{b} = \frac{\sin \tfrac{1}{2}\text{C}}{\sin(\text{A} + \tfrac{1}{2}\text{C})} = \frac{\sin \tfrac{1}{2}\text{C}}{\cos \tfrac{1}{2}(\text{A} - \text{B})},$$

or $$\text{AF} \cos \tfrac{1}{2}(\text{A} - \text{B}) = 2\text{R} \sin \text{B} \sin \tfrac{1}{2}\text{C};$$

so $$\text{BD} \cos \tfrac{1}{2}(\text{B} - \text{C}) = 2\text{R} \sin \text{C} \sin \tfrac{1}{2}\text{A},$$

$$\text{CE} \cos \tfrac{1}{2}(\text{A} - \text{C}) = 2\text{R} \sin \text{A} \sin \tfrac{1}{2}\text{B};$$

which renders the truth of the theorem manifest.

Example. By Chap. I., Prob. III., Cor. 2, we may take

$$\tfrac{1}{2}\mathrm{A} = \varphi(3), \quad \tfrac{1}{2}\mathrm{B} = \varphi(5), \quad \tfrac{1}{2}\mathrm{C} = \varphi(\tfrac{11}{3});$$

$$\sin\tfrac{1}{2}\mathrm{A} = \tfrac{3}{5}, \quad \cos\tfrac{1}{2}\mathrm{A} = \tfrac{4}{5}, \quad \sin\mathrm{A} = \tfrac{24}{25}, \quad \cos\mathrm{A} = \tfrac{7}{25};$$

$$\sin\tfrac{1}{2}\mathrm{B} = \tfrac{5}{13}, \quad \cos\tfrac{1}{2}\mathrm{B} = \tfrac{12}{13}, \quad \sin\mathrm{B} = \tfrac{120}{169}, \quad \cos\mathrm{B} = \tfrac{119}{169};$$

$$\sin\tfrac{1}{2}\mathrm{C} = \tfrac{33}{65}, \quad \cos\tfrac{1}{2}\mathrm{C} = \tfrac{56}{65}, \quad \sin\mathrm{C} = \tfrac{3696}{4225}, \quad \cos\mathrm{C} = \tfrac{2047}{4225};$$

and taking $2\mathrm{R} = \frac{4225}{24}$, we shall have

$$a = 169, \quad b = 125, \quad c = 154;$$

$$r = 41\tfrac{1}{4}, \quad r_a = 168, \quad r_b = 93\tfrac{1}{3}, \quad r_c = 140;$$

$$\mathrm{AO} = 68\tfrac{3}{4}, \quad \mathrm{BO} = 107\tfrac{1}{4}, \quad \mathrm{CO} = 81\tfrac{1}{4};$$

$$\mathrm{AO}' = 280, \quad \mathrm{BO}' = 182, \quad \mathrm{CO}' = 195;$$

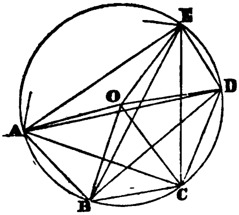
&c.

$$\mathrm{AD} = 110\tfrac{110}{279}, \quad \mathrm{BE} = 148\tfrac{244}{323}, \quad \mathrm{CF} = 123\tfrac{17}{21};$$

$$\mathrm{AF} = 65\tfrac{10}{21}, \quad \mathrm{BD} = 93\tfrac{79}{279}, \quad \mathrm{CE} = 65\tfrac{130}{323}.$$

To make these all integers it would be necessary to multiply them by 162792.

Theorem III. If the radius of the circle which circumscribes a plane polygon of any number of sides, and the trigonometrical functions of the half angles which subtend the sides at the centre of this circle, are all expressed in rational numbers; then the sides and diagonals of this polygon, its area, and the areas of the triangles into which the diagonals divide the polygon, are all expressed in rational numbers.



Demonstration. Let R be the radius of the circle, O its centre, and ABCD.... a polygon of n sides inscribed within it; let also angle AOB $= 2$A,

$$\text{BOC} = 2\text{B},$$

$$\text{COD} = 2\text{C},$$

&c.

so that
$$2\text{A} + 2\text{B} + 2\text{C} + \&\text{c.} = 360°,$$

$$\text{A} + \text{B} + \text{C} + \&\text{c.} = 180°.$$

Then it is plain from Theorem I., that we have

$$\text{AB} = 2\text{R} \sin \text{A},$$

$$\text{BC} = 2\text{R} \sin \text{B},$$

$$\text{CD} = 2\text{R} \sin \text{C},$$

&c.

$$\text{AC} = 2\text{R} \sin (\text{A} + \text{B}),$$

$$\text{AD} = 2\text{R} \sin (\text{A} + \text{B} + \text{C}),$$

$$\text{BD} = 2\text{R} \sin (\text{B} + \text{C}),$$

$$\text{BE} = 2\text{R} \sin (\text{B} + \text{C} + \text{D}),$$

&c.

For the area of the Polygon, we shall have

$$\text{P} = 2\text{R}^2 (\sin 2\text{A} + \sin 2\text{B} + \sin 2\text{C} + \&\text{c.});$$

and for the triangles into which the diagonals divide it

$$\text{ABC} = 2\text{R}^2\{\sin 2\text{A} + \sin 2\text{B} - \sin 2(\text{A}+\text{B})\}$$

$$= 8\text{R}^2 \sin \text{A} \sin \text{B} \sin(\text{A}+\text{B}),$$

$$\text{ACD} = 2\text{R}^2\{\sin 2(\text{A}+\text{B}) + \sin 2\text{C} - \sin 2(\text{A}+\text{B}+\text{C})\}$$

$$= 8\text{R}^2 \sin(\text{A}+\text{B}) \sin \text{C} \sin(\text{A}+\text{B}+\text{C}),$$

&c.;

which proves the truth of the Theorem.

Example 1. For the Quadrilateral, we may take

$$\text{A} = \varphi(3), \quad \text{B} = \varphi(5), \quad \text{C} = \varphi(\tfrac{9}{7});$$

$$\text{A}+\text{B} = \varphi(\tfrac{7}{4}), \quad \text{B}+\text{C} = \varphi(\tfrac{19}{22});$$

$$\text{A}+\text{B}+\text{C} = \varphi(\tfrac{7}{17}), \quad \text{D} = 180^\circ - \varphi(\tfrac{7}{17}) = \varphi(\tfrac{17}{7}).$$

Then we shall find

$$\text{AB} = 2\text{R}\,.\,\tfrac{3}{5}, \quad \text{BC} = 2\text{R}\,.\,\tfrac{5}{13}, \quad \text{CD} = 2\text{R}\,.\,\tfrac{63}{65},$$

$$\text{AD} = 2\text{R}\,.\,\tfrac{119}{169}; \quad \text{AC} = 2\text{R}\,.\,\tfrac{56}{65}, \quad \text{BD} = 2\text{R}\,.\,\tfrac{836}{845}.$$

Taking the diameter $2\text{R} = 845$, we have for the sides

507, 325, 819, 595;

for the two diagonals

728, 836;

and for the areas

$$\text{P} = 1123584, \quad \text{ABC} = 283920, \quad \text{BCD} = 526680.$$

Example 2. For the Pentagon, we may take

$$\text{A} = \varphi(3), \quad \text{B} = \varphi(5), \quad \text{C} = \varphi(\tfrac{7}{4}), \quad \text{D} = \varphi(8);$$

$$\text{A} + \text{B} = \varphi(\tfrac{7}{4}) \quad \text{B} + \text{C} = \varphi(\tfrac{31}{27}), \quad \text{C} + \text{D} = \varphi(\tfrac{4}{3});$$

$$\text{A} + \text{B} + \text{C} = \varphi(\tfrac{33}{56}), \qquad \text{B} + \text{C} + \text{D} = \varphi(\tfrac{17}{19}),$$

$$\text{A} + \text{B} + \text{C} + \text{D} = \varphi(\tfrac{16}{17}), \quad \text{E} = 180° - \varphi(\tfrac{16}{37}) = \varphi(\tfrac{37}{16}).$$

Then we shall find

$$\text{AB} = 2\text{R} \,.\, \tfrac{3}{5}, \quad \text{BC} = 2\text{R} \,.\, \tfrac{5}{13}, \quad \text{CD} = 2\text{R} \,.\, \tfrac{56}{65},$$

$$\text{DE} = 2\text{R} \,.\, \tfrac{18}{65}, \quad \text{EA} = 2\text{R} \,.\, \tfrac{1184}{1625};$$

$$\text{AC} = 2\text{R} \,.\, \tfrac{56}{65}, \quad \text{AD} = 2\text{R} \,.\, \tfrac{3696}{4225}, \quad \text{BD} = 2\text{R} \,.\, \tfrac{837}{845},$$

$$\text{BE} = 2\text{R} \,.\, \tfrac{323}{325}, \quad \text{CE} = 2\text{R} \,.\, \tfrac{24}{25}.$$

Taking the diameter $2\text{R} = 21125$, we have for the sides

12675, 8125, 18200, 5200, 15392;

and for the diagonals

18200, 18480, 20925, 20995, 20280;

the areas, which will be expressed in whole numbers, may easily be calculated.

Cor. It is obvious enough, although it has not been judged necessary to include it in the body of the theorem, that in the hypothesis of the theorem, the segments into which the diagonals divide each other will also be expressed in rational numbers :—for the angles made by any pair of them will be expressed by the sums or differences of two or more of the angles A, B, C, &c.

THEOREM IV. *If the radius of the circle which circumscribes a plane polygon of any number of sides, and the trigonometrical functions of one-fourth of the angles which subtend the sides at the centre of this circle, are all expressed in rational numbers; then the radii of the circles which touch any three of the sides and diagonals of this polygon, the lines joining the angular points of the polygon with the centres of these circles, the lines bisecting the angles made by any pair of the sides and diagonals of the polygon, and terminating in any other side or diagonal, &c., &c., will be expressed in rational numbers.*

The truth of this Theorem is sufficiently obvious from the Demonstration of the two preceding ones.

PROBLEM I. *To find parallelograms whose sides, diagonals, and area, are integers.*

Lenhart, Math. Miscellany, Quest. 119.

Solution. Let a, b be the two adjacent sides of the parallelogram; A the angle included between them; c, d its two diagonals; and P its area. Then

$$a^2 + b^2 - 2ab \cos A = c^2,$$

$$a^2 + b^2 + 2ab \cos A = d^2,$$

$$P = ab \sin A.$$

By adding and subtracting the two first, we have

$$2a^2 + 2b^2 = c^2 + d^2,$$

$$4ab \cos A = d^2 - c^2.$$

The first of these equations is solved, as we have seen, by taking

$$c = (a + b) \cos B + (a - b) \sin B,$$

$$d = (a + b) \sin B - (a - b) \cos B;$$

and these substituted in the second, give

$$2ab \cos A = (b^2 - a^2) \sin 2B - 2ab \cos 2B.$$

Now put $$\frac{b}{a} = \cot \tfrac{1}{2}C,$$ that is

$$\frac{2ab}{\sin C} = \frac{b^2 - a^2}{\cos C} = b^2 + a^2;$$

then this equation becomes

$$\sin C \cos A = \sin (2B - C);$$

thence we have

$$\sin C \cos^2 \tfrac{1}{2}A = \sin B \cos (C - B),$$

$$\sin C \sin^2 \tfrac{1}{2}A = \cos B \sin (C - B),$$

$$\sin^2 C \sin^2 A = \sin 2B \sin 2(C - B);$$

which is the same as equation (*c*) of Problem XXX., Chapter II. If we take, as in the second solution of that equation,

$$\tan \tfrac{1}{2}B = \tfrac{1}{2}(1 + 3 \cos \theta), \qquad \tan \tfrac{1}{2}(C - B) = \tfrac{1}{2}(1 - 3 \cos \theta);$$

we get

$$\tan \tfrac{1}{2}C = \frac{4}{3 + 9 \cos^2 \theta} = \frac{a}{b};$$

and therefore we may take

$$a = 4t, \qquad b = 3t (1 + 3 \cos^2 \theta).$$

Then we shall find

$$c = t(3 \cos \theta - 1)(5 + 3 \cos \theta), \quad d = t(3 \cos \theta + 1)(5 - 3 \cos \theta);$$

$$P = ab \sin A = \tfrac{1}{2}(a^2 + b^2) \sin C \sin A = 12t^2 \sin \theta (9 \cos^2 \theta - 1).$$

Example. Let $\theta = \varphi(2)$, $t = \frac{25}{4}$; we have

$$a = 25, \qquad b = 39;$$

$$c = 34, \qquad d = 56;$$

$$P = 840.$$

The above equation has also the obvious solutions

$$C = 2B, \qquad A = 90^\circ;$$

when the parallelogram is rectangular; and

$$C = 90^\circ, \qquad A = 2B;$$

when it is equilateral.

PROBLEM II. To find the sides of right angled triangles, in rational numbers, which have equal areas.

Solution. Let h, h' be the hypothenuses of two right angled triangles; A, B their acute angles; then

$$a = h \sin A, \qquad b = h \cos A;$$

$$a' = h' \sin B, \qquad b' = h' \cos B;$$

are the legs of these triangles; and, when their areas are equal, we must have

$$\tfrac{1}{4}h^2 \sin 2A = \tfrac{1}{4}h'^2 \sin 2B,$$

$$\frac{h^2}{h'^2} = \frac{\sin 2B}{\sin 2A};$$

an equation, the solution of which is evidently dependent on that of equation (c) Prob. XXX., Chap. II.

If we take, as in the first solution of that equation,

$$\tan \tfrac{1}{2}B = \sin 2A\,;$$

we shall find

$$\left.\begin{aligned} h' &= h\,.\,\frac{1+\sin^2 2A}{2\cos 2A} = \frac{h^4+4a^2b^2}{2h(b^2-a^2)},\\ a' &= h\tan 2A \qquad\;\;= \frac{2abh}{b^2-a^2},\\ b' &= \tfrac{1}{2}h\cos 2A \qquad = \frac{b^2-a^2}{2h}, \end{aligned}\right\} \;\cdot\;\cdot\;(a).$$

By means of these formulas, any number of triangles, having the same area, may be deduced from each other by successive substitution.

Example. If $h=5$, $a=3$, $b=4$, we get

$$h' = \tfrac{1201}{70}, \qquad a' = \tfrac{120}{7}, \qquad b' = \tfrac{7}{10}\,;$$

and by writing these values of h', a', b' instead of h, a, b, we get another triangle having the same area, &c.

Again, in the third solution of equation (c), Prob. XXX., this solution is made dependent on the equation

$$\tan^2 \tfrac{1}{2}A + \tan \tfrac{1}{2}A \tan \tfrac{1}{2}B + \tan^2 \tfrac{1}{2}B = 1,$$

which may be put in the form

$$(2\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B)^2 + 3\tan^2 \tfrac{1}{2}B = 4,$$

and therefore, by Prob. IV., Example 2, Chap. II.,

$$2\tan \tfrac{1}{2}A + \tan \tfrac{1}{2}B = \pm\, t(\cos^2 \tfrac{1}{2}\theta - 3\sin^2 \tfrac{1}{2}\theta),$$

$$\tan \tfrac{1}{2}B = t\sin\theta, \qquad 2 = t\,(\cos^2 \tfrac{1}{2}\theta + 3\sin^2 \tfrac{1}{2}\theta\,;$$

thence we have, by eliminating t,

$$\tan\tfrac{1}{2}B = \frac{2\sin\theta}{2-\cos\theta}, \qquad \tan\tfrac{1}{2}A = \frac{-\sin\theta \pm 2\cos\theta \mp 1}{2-\cos\theta};$$

$$\frac{h}{h'} = \frac{\cos^2\tfrac{1}{2}B}{\cos^2\tfrac{1}{2}A} = \frac{1+\tan^2\tfrac{1}{2}A}{1+\tan^2\tfrac{1}{2}B} = \frac{(2-\cos\theta)^2+(1-2\cos\theta\pm\sin\theta)^2}{8-4\cos\theta-3\cos^2\theta}.$$

Then we may take

$$h = t\{(2-\cos\theta)^2 + (1-2\cos\theta\pm\sin\theta)^2\},$$

$$h\sin A = 2t(2-\cos\theta)(-\sin\theta\pm 2\cos\theta\mp 1),$$

$$h\cos A = t\{(2-\cos\theta)^2 - (1-2\cos\theta\pm\sin\theta)^2\};$$

$$h' = t(8-4\cos\theta-3\cos^2\theta),$$

$$h'\sin B = 4t\sin\theta(2-\cos\theta),$$

$$h'\cos B = t\cos\theta(5\cos\theta-4).$$

These formulas give three triangles having equal areas, for the same value of θ.

Example. If $\theta=\varphi(2)$, $t=25$, we have

$$\begin{array}{rrr} 58, & 42, & 40; \\ 74, & 70, & 24; \\ 113, & 112, & 15, \end{array}$$

for the sides of three triangles, each of whose areas is 840. If we write these separately in the formulas (a), we shall get

$$\frac{1412881}{1189}, \quad \frac{48720}{41}, \quad \frac{41}{29};$$

$$\frac{2579761}{39997}, \quad \frac{62160}{1081}, \quad \frac{1081}{37};$$

$$\frac{174336961}{2784094}, \quad \frac{379680}{12319}, \quad \frac{12319}{226},$$

1,28

2,00